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Analysis of Student Learning Outcomes in Proving Trigonometric Identities from Problem Based Learning Class

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Abstract: The topic of trigonometric identity verification given by researchers through problem-based learning in a high school in Yogyakarta has provided opportunities for students to conclude that in solving trigonometric identity verification problems three strategies can be used. This research aims to determine the learning outcomes of students in solving trigonometric identity verification questions after the learning. Observation of learning outcomes is done by analyzing the abilities and failures of students based on Indicators of Achievement of Competence. The research conducted was qualitative research with data obtained through give a 15-minutes test containing two problems, which given to students, and interviews to three students chosen based on variations in different answers. The results obtained are from the three students, only one student can solve all the problems. Two other students were unable to complete those problems, both the first and the second. The reason for this is not because the two students did not understand how to prove trigonometric identity, but because the two students were less skilled in performing basic algebraic operations related to fraction forms.

Keywords: learning outcomes; proving; strategies; trigonometric identities; problem-based learning.

Introduction

The fact that occurred in one of the high schools in Yogyakarta to students in the two classes of grade 10 in the 2016-2017 school year showed that the learning outcomes of students on the topic of trigonometric identity verification were unsatisfying for teacher. Based on the evaluation of learning outcomes in the form of daily tests, from 35 Math and Science students in grade 10-6 there were 19 students who did not pass. In grade 10-7, the number of students who did not pass the evaluation was more, namely from 34 Math and Science students there were 28 students who did not pass. The failure of students in the daily test can also be seen from the average test scores of the trigonometric identity of the two classes. The average score of grade 10-6 is 71, while the average score of grade 10-7 is 64.

On March 21 to April 8, 2018, researchers tried to teach the topic of trigonometric identity verification in one class in the same high school using Problem Based Learning based on the Hypothetical Learning Trajectory (HLT) that had been created. The purposes of learning listed in HLT are that students can (1) find strategies to prove trigonometric identity and (2) prove trigonometric identity.

To find out whether the learning goal is achieved or not, the researchers set The Indicator of Competency Achievement (ICA): Students can prove trigonometric identity if (1) students can choose strategies to prove trigonometric identity and (2) students are able to apply the chosen strategy correctly.

One of the results obtained from the learning is that students are able to draw conclusions about strategies that can be used to solve trigonometric verification problems; namely (1) decipher the right side to obtain the mathematical expression equal to the left side, (2) decipher the left side to obtain the mathematical expression equal to the right side, and (3) decipher the right and left sides together so that the mathematical expressions are equal. Before reaching these conclusions there are other strategies that are also used by students; namely (1) visualize with a right triangle image and then analyze analytically (with certain variables or/and directly with certain numbers) and (2) use direct examples using certain special angles.

Research related to the verification of trigonometric identities was also carried out by Huljannah et al. (2015), who found that students could make mistakes when proving trigonometric identities. The mistakes in solve problems include improper procedures and skills hierarchy problems. On the other hand, Stefanowicz (2014: 32) states that students can make mistakes in doing mathematical proof. The mistakes made include misunderstanding of definitions, not enough words, lack of understanding, and incorrect steps.

Based on the description above, researchers are interested in analyzing the learning outcomes of students in solving trigonometric identity verification problems. The aim of this research is to determine the ability of students in solving trigonometric identity verification problems. Therefore, researchers asked questions about how student learning outcomes in solving trigonometric identity verification problems after being given problem-based learning on the topic of trigonometric identity verification.

Problem Based Learning, Proof in Mathematics Learning, and Trigonometric Identity Verification

Linda Torp and Sara Sage (in Rusman, 2016) point out that problem-based learning is focused, experiential learning (minds-on, hands-on) organized around the investigation and resolution of messy, real-world problem. PBL—which incorporates two complementary processes, curriculum organization and instructional strategy—includes three main characteristic, namely (1) engages students as stakeholders in a problem situation, (2) organizes curriculum around a given holistic problem, enabling students learning in relevant and connected ways, and (3) creates a learning environment in which teachers coach student thinking and guide student inquiry, facilitating deeper levels of understanding.

A proof is a sequence of logical statements, one implying another, which gives an explanation of why a given statement is true (Stefanowicz, 2014). Mathematical proof is a formal way to express certain types of reasoning and justification (NCTM, 2000). By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content, students can see and think that mathematics gives meaning. The purpose of standard reasoning and proof in mathematics learning according to NCTM (2000) is that students can (1) view reasoning and proof as fundamental aspects of mathematics, (2) create and investigate mathematical conjectures, (3) develop and evaluate arguments and mathematical proofs, and (4) choosing and using various kinds of reasoning and proof methods.

In proving the identity of trigonometry there are several types of methods. Some methods of verifying trigonometric identities include using fundamental trigonometric identities (Abramson, 2017); namely Pythagoras identity, odd-even identity, opposite identity, and comparative identity.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the Pythagorean identities, the even-odd identities, the reciprocal identities, and the quotient identities.

Research Methods

This research was a qualitative research according to Moleong (2009), whose purpose was to understand the phenomenon of what was experienced by the subject of research with the phenomenon was described in the form of words. The subjects observed were three students from one of grade 10th classes in one of high schools, in Yogyakarta. The selection of the three students as subjects was conducted based on variations in different answers. Data collection was carried out, in addition to giving 15-minutes test problems, interviews were also conducted with the three subjects for the answers given. Data analysis techniques are based on data analysis techniques according to Miles and Huberman (Sugiyono, 2014); namely data reduction, data presentation, and conclusion drawing.

Results and Discussion

Analysis of students' answers is done based on the ICA that has been created; i.e. students can prove trigonometric identity if students (1) can choose a strategy to prove trigonometric identity and (2) be able to apply a correctly chosen strategy. The ICA is then described into the problem indicators. The ICA is that students are categorized as fulfilling trigonometric identity verification competencies if: (1) Students can choose a strategy to prove trigonometric identity. For example, to prove $\sin\alpha + \cos\alpha\cot\alpha = \csc\alpha$ students can choose the strategy of deciphering the left side until the results of the right side are obtained. (2) Students are able to use basic identities to solve problems. For example, students can use a basic identity $\frac{\sin\alpha}{\cos\alpha}$ to replace the left side equal to $\tan\alpha\sin\alpha+\cos\alpha=\left(\frac{\sin\alpha}{\cos\alpha}\right)\sin\alpha+\cos\alpha$ and so on. (3) Students can operate mathematically one or both sides so that the proof of trigonometric identity is obtained. For example, the left side equal to $\tan\alpha\sin\alpha+\cos\alpha=\frac{\sin\alpha}{\cos\alpha}\sin\alpha+\cos\alpha=\frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha}=\frac{1}{\cos\alpha}=\sec\alpha$ equal to right side. Students are said to not fulfill the ICA if they cannot fulfill one of the indicators of the problem. The following is a 15-minutes test problem given to students.

Prove that
(1) $\sin\alpha+\cos\alpha\cot\alpha=\csc\alpha$
(2) $\tan\alpha=\frac{\sin\alpha+\tan\alpha}{1+\cos\alpha}$

The Answers of Problem Number 1

The following are described answers to problems number 1 given by three students named Vian, Dodi, and Maman (all three names are not real names).

1. *Vian's answer to Problem Number 1*. At the beginning of the interview Vian said that he knew three strategies for solving trigonometric identity verification problems; namely deciphering the left side until the result is equal to the right side, deciphering the right side until the result is equal to the left side, and decipher the left and right sides together until the equal result is obtained. Besides that, Vian also said that there was another strategy that could be used, namely using a help triangle.
   In the answer to the first problem Vian uses the strategy of deciphering the left side to get the same result as the right side. The reason for choosing the strategy is because according to him the left side contains complicated mathematical expressions so that it can be simplified. In addition, it is also seen that Vian uses Reverse Identity like $\cos\alpha=\frac{1}{\sec\alpha}$ and $\cot\alpha=\frac{1}{\tan\alpha}$ and Pythagoras Identity like $\sin^2\alpha+\cos^2\alpha=1$. Until this stage, Vian had fulfilled the first and second problem indicators. By doing algebraic manipulation, Vian can find that from the results of the decomposition of the left side is the same as the mathematical expression on the right side. From the results of the breakdown, Vian can also conclude that the statement given to problem number 1 proved to be true. This stage shows that the third problem indicator is fulfilled.

Interviewer	*This number one, what strategy do you use? Can you explain how to solve it?*
Vian	*This, I outline the left section so that the left segment is the same as the right segment. So, sin alpha plus cos alpha multiplied by cotan alpha, it is to be equal to cosec alpha.*
Interviewer	*Why on this matter do you decipher the left? Why not right?*
Vian	*Because it is complicated and can be simplified.*

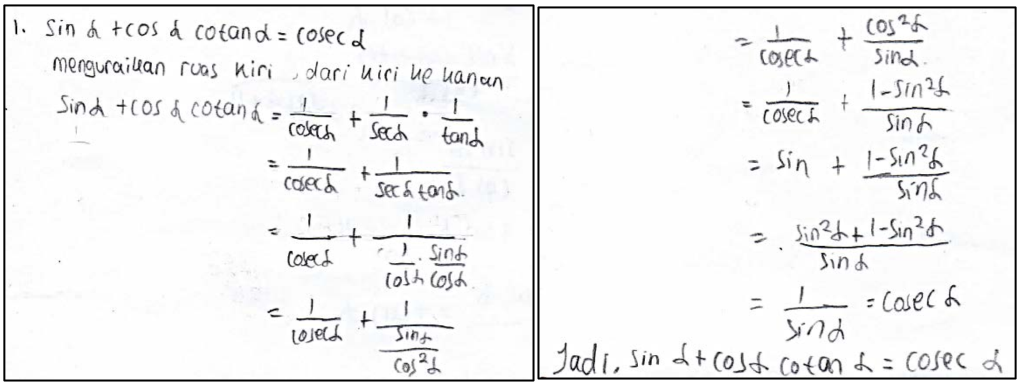

Figure 1. Vian’s answer to question Number 1

Even so it is seen that to obtain $\csc\alpha$, Vian changed form $\sin\alpha$ into form $\frac{1}{\csc\alpha}$. However, after a few steps Vian changed $\frac{1}{\csc\alpha}$ back to being $\sin\alpha$. This shows that the algebraic manipulation process undertaken by students in solving the trigonometric identity verification questions for number 1 also involves a trial and error process.

Interviewer	*Why do you change this ($\sin\alpha$) to cosec?*
Vian	*Because I'm using the opposite formula, sir. Because I think, this will result in cosec, so I change it to cosec. This is back again, sir. Hehe. Because this is a sin, I think it's easier to turn it around again.*

Part of Vian's answer to number 1 shows that students are able to fulfill all three problem indicators. Thus, the ICA for number 1 is also fulfilled so that it can be said that for Vian the second HLT goal for number 1 has been reached.

2. *Dodi's answer to Problem Number 1.* At the beginning of the interview, Dodi said that there were three strategies that could be used to solve trigonometric verification questions. Dodi's strategy is to use aid triangles which are then analysed analytically, using three strategies to decipher a complicated mathematical expression into a simpler mathematical expression (called by Dodi with other terms or names of sine, cosine, and tangent), and use direct evidence with certain numbers.

Interviewer	*How many types of strategies do you know to prove?*
Dodi	*I think there are three, sir. Which uses x y x y, triangle. Then this is, uses numbers.*
Interviewer	*How is this?*
Dodi	*Here, I use sin cos tan which is another name.*

In the process of proof, Dodi chose to use the strategy of deciphering the left section until the same result was obtained with the right segment. The reason for choosing the strategy is because according to him the left side contains complicated mathematical expressions compared to the right side so that it can be simplified. In the answer it can be seen that Dodi was able to use a basic identity in performing algebraic manipulations to prove the statement given so as to obtain evidence that the left side is the same as the right segment.

But at the end of the proof, Dodi was not complete in giving an evaluation of the results of proof. The truth of the statement given is only expressed by students through the word “TERBUKTI” (PROVEN). This does not provide an explanation of the truth of the statement given to the question explicitly. Even so in the interview, Dodi can provide an evaluation of the results of the evidence.

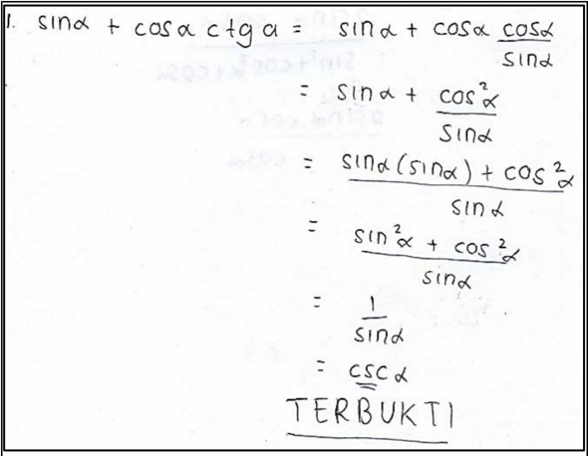

Figure 2. Dodi’s answer to question number 1

Interviewer	*Then, number one? Try to explain your answer.*
Dodi	*This is cotangen, it can be changed to cos per sin, sir. Then cos times cos so it's cos squared. Then let it be added, we equate the denominator, sir. So sin times sin plus cos squared per sin. Let sin be squared per cos squared per sin. Sin squared plus cos squared equal to one. So one sin, same as cosecan teta. So it is proven that this (pointing to the left hand side of the question) is the same as this (pointing to the right side of the question).*
Interviewer	*So you decipher this (pointing to the left side)?*
Dodi	*Yes. The left side.*
Interviewer	*Why are you using this strategy?*
Dodi	*Because I mastered this strategy the most, sir.*
Interviewer	*If what is deciphered right, can it or not?*
Dodi	*You can, but the way is more complicated, sir. If the left side is easier to break.*
Interviewer	*More complicated where are the left and right segments?*
Dodi	*The left one, sir. So it can be simplified more easily.*
Interviewer	*This is where, how come there are sin times sin?*
Dodi	*It's added, so the denominator is equated. So sin multiplied by sin divided by sin, so that together sin.*

From Dodi's answer, it was seen that students were able to fulfill the three problem indicators. Thus, for Dodi in number 1, the ICA is fulfilled and the goal of the second HLT is achieved.

3. Maman’s answer to Problem Number 1. At the beginning of the interview, Maman said that he knew three strategies for solving trigonometric identity verification questions. The three strategies that Maman refers to are outlining the left section so that the same result is obtained with the right segment, describing the right segment so that the same result is obtained with the left segment, and describing the two segments so that the same result is obtained. In addition, Maman also said that there are other strategies that can be used, namely using a help triangle which is then analyzed analytically.

   In solving question number 1, Maman deciphered the left and right segments together so that the same results were obtained. The reason for choosing this strategy is because Maman considers that the two segments are complicated, so they need to be simplified. From Maman’s answer, it can be seen that students can apply their strategies correctly as indicated by their ability to perform algebraic manipulations accompanied by the use of the correct basic identity so that evidence can be obtained. Until that stage, students have fulfilled all three problem indicators. The results of the interview also show that students can draw conclusions about the truth of statement number 1. However, in the settlement the students do not explicitly provide conclusions about the truth of the statement given to the question. Nevertheless, the truth of the statement given is shown by giving an expression in the form of a symbol of truth. Thus, for question number 1, Maman has fulfilled the ICA and the second HLT goal is reached.

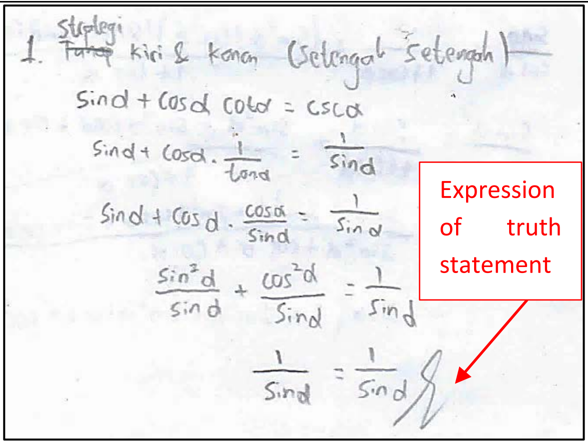

Figure 3. Maman's answer to question Number 1

Interviewer	*From this problem, what were you told?*
Maman	*Ask to prove that on the left hand side you can get the right side.*
Interviewer	*Do you know how many strategies to prove this?*
Maman	*The first can be from left to right. So the left side is manipulated. The second from right to left. Right manipulated. Otherwise it means half-half, both are manipulated.*
Interviewer	*Try to explain how you found this.*
Maman	*(Maman explains step by step).*
Interviewer	*What can you conclude?*
Maman	*Yeah, if this is the case, it means that the left side is the same as the right side. Because the results are the same.*
Interviewer	*What is the reason for manipulating the two segments?*
Maman	*How about ... For example eight subtract seven equals one. One can be changed to any value. For example, five subtracts four. You can, too. Well, what can be simplified is the complicated one, the left side. The right one changes a little, giving capital is small, simplified.*
Interviewer	*So do you think everything is complicated?*
Maman	*Em ... it's more complicated on the left.*

From the answers to questions number 1 above, the three students have met the three problem indicators. Therefore, for question number 1, the three students have fulfilled the ICA and the second HLT goal has been achieved, noting that one student also uses trial and error when doing algebraic manipulation. That is, students make mathematical conjectures with trial and error strategies. In addition, from the three students there are two students who are less complete in giving an evaluation of the results of the evidence obtained from the resolution so that it can be said that for question number two, two students are less able to evaluate mathematical proofs.

The Answers of Problem Number 2

The following are outlined answers to the number 2 questions given by the three students.

1. *Vian's answer to Problem Number 2.* In answer number 2, students use a strategy outlining the right segment to obtain the same shape as the left side. When interviewing students also explained that the strategy was chosen because the right section contained complicated mathematical expressions that needed to be simplified. At that stage the students have fulfilled the first indicator of the problem.

Interviewer	*Number 2, how? What strategy do you use?*
Vian	*If this, I decipher the right section so that the result is the same as the left section.*
Interviewer	*Why are you deciphering the right side? How come it's not the left side?*
Vian	*Because the right is more complicated, so that can be simplified.*

Seen in the answer, Vian's next process of completion was to use comparative identities such as $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$. This shows that students have fulfilled the second indicator. By doing algebraic manipulation, Vian found that mathematical expressions on the right segment are the same as mathematical expressions on the left side. At that stage, it can be said that Vian has fulfilled the third indicator of the problem. At the end of the settlement, Vian concluded that $\frac{\sin\alpha+\tan\alpha}{1+\cos\alpha} = \tan\alpha$. That is, Vian was able to provide an evaluation of the results of the evidence made.

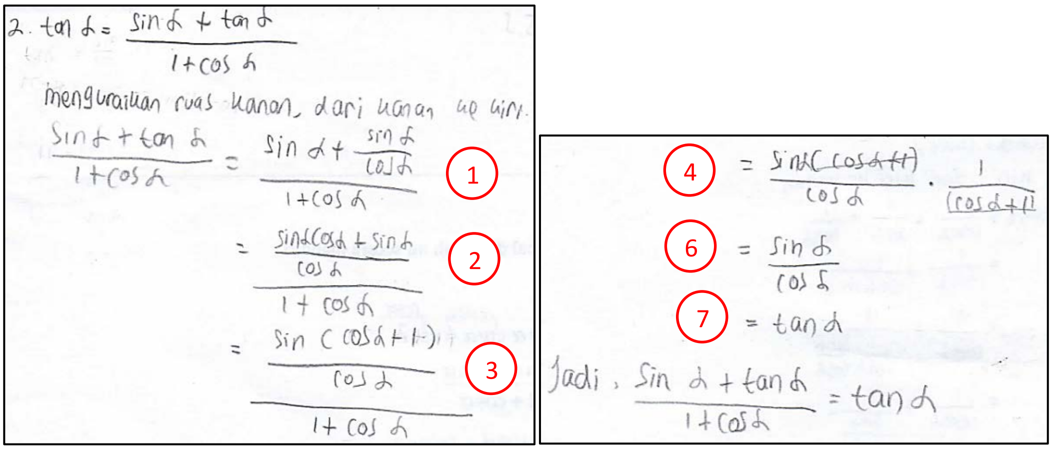

Figure 4. Vian's answer to question Number 2

Although Vian has fulfilled the three indicators of the problem, there are inaccuracies in writing the notation or symbol "equal to" (clearly seen in the second and third steps). In the answers of students, the symbol is not exactly parallel to the main fraction symbol, but rather parallel to the numerator fraction symbol.

From Vian's answer to question number 2, because students are able to fulfil all three problem indicators, the ICA is also fulfilled so that it can be said that for Vian the second HLT goal has been achieved.

2. *Dodi's answer to Problem Number 2.* In the answer to question number 2, Dodi uses a strategy to decipher the right segment to become a simpler form. Dodi also uses a basic identity to solve problems but has difficulty in the middle of the road. Seen in Dodi's answer in the third step, students make mistakes related to algebraic manipulation. This error is indicated by crossing out *cos α* which violates the rules of algebra.

Interviewer	*Why did you choose this strategy, outline the right side?*
Dodi	*The problem is that the right is more complicated, sir. So it can be simplified.*

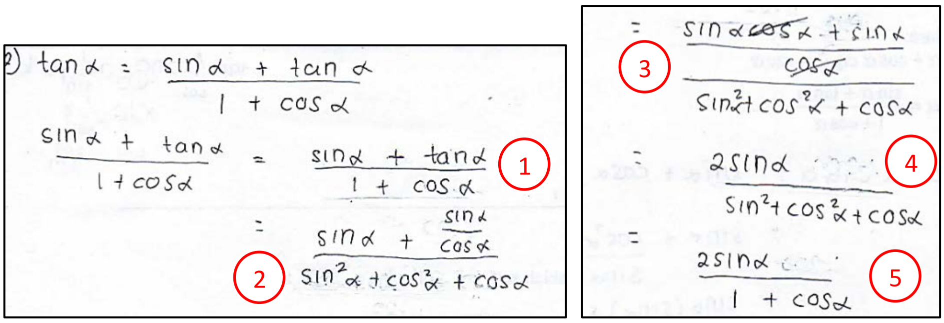

Figure 5. Dodi's answer to question Number 2

Interviewer	*This number two, how?*
Dodi	*I haven't solved it yet, sir.*
Interviewer	*How do you do it?*
Dodi	*What I have done is, sir. This changes the right side. I changed the tan, sir. So sin per cos. The bottom one, this one, I'm describing to be sin squared plus cos squared, per this. This (pointing to the third step) is wrong here, sir. The upper one can be added together, equalized by the denominator. So sin cos plus sin divided by cos, divided by sin squared plus cos squared ... Well, it should not be crossed out, but I crossed it out. So wrong.*
Interviewer	*Do you think you can make the right one? Try to do it again starting from this, which should not be crossed out earlier.*
Dodi	*(Trying to work again according to the instructions of the researcher)*

In further investigation, through interviews and improvements made by students, Dodi turned out to have a basic algebraic operating ability that was not strong enough to make mistakes as mentioned above. Therefore, the researchers tried to provide scaffolding so that eventually the students were able to answer correctly.

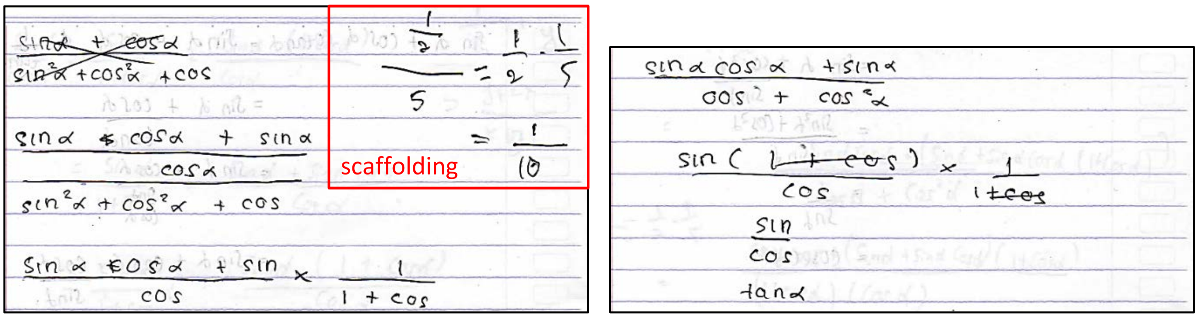

Figure 6. Dodi's answer to question Number 2 after being given a scaffolding

Dodi	*I'm confused, sir.*
Interviewer	*How confused?*
Dodi	(tried to think again for some time and didn't seem to know what to do).
Interviewer	*If there is one by two per fifth. What does this change per what, how?*
Dodi	*This (5) is raised, sir. (thinking by trying to apply it to the problem). Confused, sir.*
Interviewer	*Look at this again (pointing* $(1/2)/5$*), while writing* $\frac{1}{2}\cdot\frac{1}{5}$*). From this, how?*
Dodi	*This is one tenth, sir.*
Interviewer	*Okay. Can this be applied in the matter of no?*
Dodi	*This means ... (trying to think and not being able to immediately apply the scaffolding to the problem).*
Interviewer	*Try this sin alpha cos plus alpha sin can be changed no?*
Dodi	*Oh ... you can, sir. This should be (pointing cos times one plus cos) don't multiply first, sir? (continuing to finish) This is meeting ... the solution is like this, sir.*
Interviewer	*So what conclusion?*
Dodi	*Em ... So the conclusion is proven ... sin but plus tan per one plus cos the teta is the same as tan teta.*
Interviewer	*Okay. This is alpha.. Not teta.*
Dodi	*Oh yeah, sir.*

Another thing that attracts the attention of researchers besides the things mentioned above, it turns out that Dodi also does not memorize the names of mathematical symbols. This can be seen from the results of interviews with students who refer to the alpha symbol as a symbol of teta.

From Dodi's answer, the problem indicators that are fulfilled are only the first indicator. The second indicator has also been carried out through trial and error, but in carrying out algebraic operations it cannot carry out correctly. Thus, for question number 2, it can be said that Dodi did not fulfill the ICA and the second HLT goal was not reached.

3. *Maman's answer to Problem Number 2.* In solving question number 2, Maman chose a strategy to decipher the left and right segments at the same time. In the answer it appears that Maman has difficulty in the third step where despite difficulties, students continue to complete the solution until the seventh step. In the seventh step, students seem unable to continue the settlement.

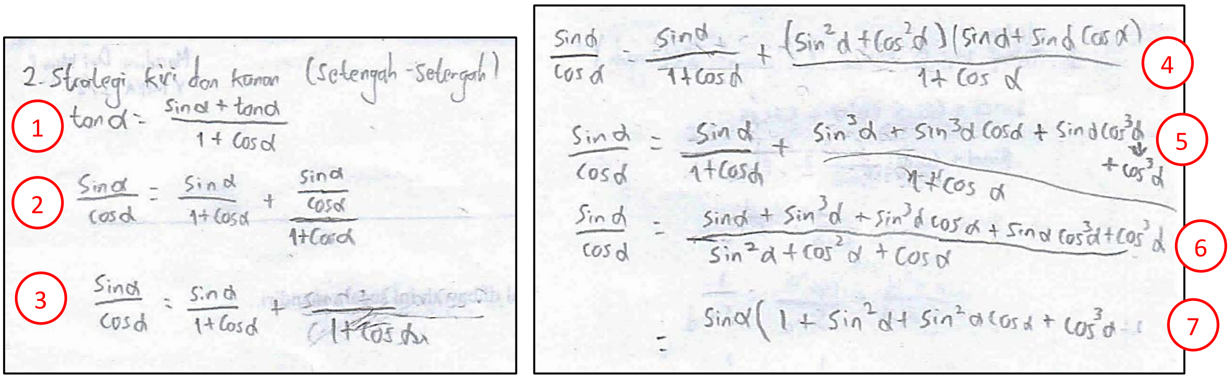

Figure 7. Maman’s answer to question Number 2

Interviewer	*Number two, how?*
Maman	*Wrong.*
Interviewer	*How is that wrong?*
Maman	*Confused how to release this plus sign.*
Interviewer	*What strategy do you use?*
Maman	*Left and right. Only the problem was, releasing one plus cos alfa. That's what makes me confused.*

In further investigation to find out where the students were wrong, interviews were conducted. It turns out that Maman is the same as Dodi. Learners make mistakes in basic algebraic operations related to the fractions deciphered by saying that it is difficult for him to decipher the denominator $(1 + \cos \alpha)$. Then educators ask students to try to correct their mistakes.

When correcting the third step, students continue to experience difficulties so that educators provide scaffolding as follows.

Interviewer	*Half divided by five, what is the result?*
Maman	*five by two.*
Interviewer	*If it's half divided by two, what is the result? You have half bread, divided by two, how much?*
Maman	*A quarter.*
Interviewer	*Well, if one per x plus two per y, what is the result?*
Maman	*Y add two x per x y.*
Interviewer	*Now, try to apply it to the problem.*

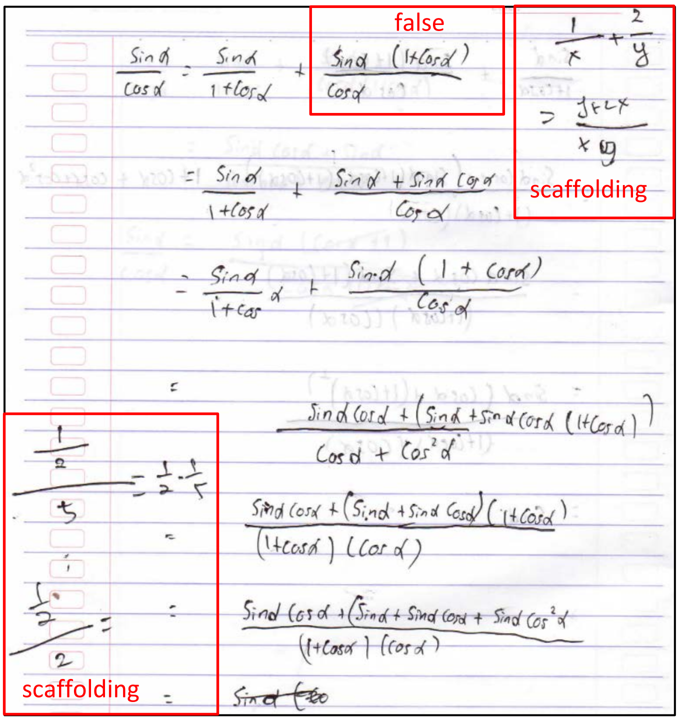

Figure 8. Maman's answer to question Number 2 after trying to do it again but is still wrong so it is given a scaffolding

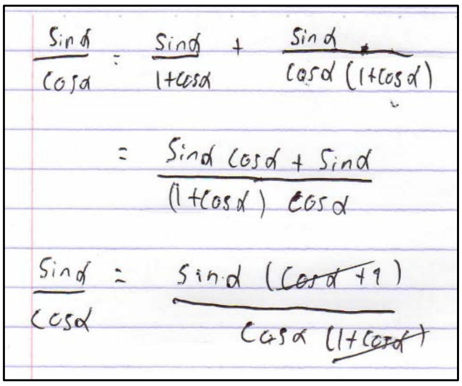

Figure 9. Maman's answer to question Number 2 is correct

By giving the scaffolding, students can finally correct the mistakes in the third step and continue to resolve.

From Maman's answer, it can be seen that the third problem indicator is not fulfilled. Therefore, it can be said that for Maman for question number 2, the ICA was not reached so that the second HLT goal for Maman was also not achieved.

From the answers to questions number 2 above, of the three students, two students could not fulfill the third indicator of the problem so that they did not fulfill the ICA. A common cause that results in an unmet CPI is that students experience difficulties related to algebraic manipulation, especially in the form of fractions. Therefore, for question number 2, the overall goal of the second HLT for the three students was not achieved.

Conclusion

Based on the descriptions above, the number 1 question can be solved by the three students while the second question can only be solved by one student. Thus, it can be concluded that of the three students selected as research subjects only one student fulfilled the ICA for both trigonometric identity verification questions. Therefore, it can be said that the second HLT goal of learning about the verification of trigonometry identity through Problem Based Learning for two students is not achieved.

A common cause that results in two students' ICAs not being fulfilled in both questions is not because students do not understand how to prove trigonometric identity, but because students have the ability to be less skilled in performing basic algebraic operations, especially algebra related to fractional forms.

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