SOME ASPECTS ON STUDENTS' MATHEMATICAL REASONING IN EXPLORING GROUP THEORY

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Abstract

Mathematical reasoning can be used to understand mathematics thinking process as axiomatic system. Mathematical reasoning can identify students' logical thinking and mathematical creativity. There are two aspects in mathematical reasoning i.e. formal and informal aspects. The students' mathematical reasoning can be examined by using mathematical tasks in which the students justify and communicate mathematics ideas. The aims of this study are to describe 1) formal aspect of students' mathematical reasoning and 2) informal aspect of students' mathematical reasoning when they solved the tasks. The mathematical tasks used in this study are on the topics group theory. We found that the students still lack reasoning in explaining the connection on premises that were used in formal proof. The power of students' intuition also still low in choosing the best strategy to solve the problem.

Keywords: mathematical reasoning, intuition, mathematical task, group theory.

Introduction

Group theory is one of the topics in modern algebra. Characteristics Group theory emphasizes on the abstract thinking aspect. Classical algebraic concepts at the secondary school level are generalized through abstraction. This process require more logical thinking than arithmetic skills. The ability to think logically involves more mental activity to reason, justifying and verifying mathematical ideas.The most popular illustration is basic of operations and the cardinality of sets. The structure of all set of integer with addition is group. By the property of inverve element, the notion of substruction comes later. Eventually, the group structure of nonzero real number with multiplication introduces us the notion of division. So, the students are intoduced with the notion of addition, substruction, multiplication and division by abstract concept in group theory. The next illustration about cardinality of sets. As we know in the concept of finite set, two sets have same cardinality if and only if the number of their element are equal. Something that litle difference occours in infinite sets. Consider the set of all integer number $Z = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$. If we take all the even number then we have $E = \{..., -4, -2, 0, 2, 4, ...\}$. Now, we have set E is a subset of set and make sense that the number of E is less then . But, the last statement is not true that set E and have difference cardinality. The student must generalize the definition of the sets cardinality. This concept in group theory is called isomorphism.

The provious illustrations show us that the student's ability to give reasons in every argument becomes very important in the abstraction process. Supporting statements they can use to justify a mathematical idea. These arguments can be either abstract or factual concepts directly related to the student's experience. The process of building the ability to provide a logical reason for concluding mathematical ideas is one of the main objectives of learning activities in the topic of group theory. The term of mental activities in reasoning skill for mathematical ideas in this paper is mathematical reasoning.

The mental activity of reasoning, justification and verification of mathematical ideas begins after students understand the problem. Students start to think about how strategies to solve the problem and then execute in the form of formal proof. Selection of strategies to solve problems is not always easy. The strategy chosen by the students was not always able to find the solution of the problem. The selection of an efficient strategy depend on each student's experience in doing something similar previously. The choice of strategy usually appears subjective, direct and specific. Furthermore, students provide formal arguments to solve the existing problems. This argument can either be written or expressed by providing logical reasons. The mental process in determining the strategy in solving mathematical problems is an informal aspect in mathematical reasoning. The activity is still very subjective and suddent. Meanwhile, the activity in giving arguments logically is a formal aspect in mathematical reasoning. These two aspects determine the ability of students to provide reason, justification and verification of abstract concepts in mathematics.

Abstract mathematical problems are given to students to know the extent of students' mathematical reasoning. These concept (problems) can be assembled in mathematical tasks. These tasks are constructed in such a way as to accommodate all students' abilities. The tasks are designed by considering the number of strategies that can be used to solve the problem, visual interpretation of the problem, connectivity in procedures and quality of explanation requirement.

Based on the above explanation, this paper will point out the description of aspects of students' mathematical reasoning abilities through an analysis of their reasons presented in writing to solve their problems and responses in the interview.

Theory

A. Aspects of Mathematical Reasoning

Mathematical reasoning is as vehicle to justify any mathematical ideas. This ability is needed by students considering that mathematics is a mental activity. Each mathematical idea is connect to other ideas, requiring an ability to verify the ideas. Students' mathematical reasoning involves the informal and formal aspects.

1. Informal Aspects of Mathematical Reasoning

Before ideas, concepts, and mathematical proofs are formally presented, there is first a mental process in the framework of an initial understanding of the problem. This initial process is very important for students to find the most suitable strategy to solve the problem or to relate the concepts related to the concept being studied. This activity is highly subjective. Each student will have varying views on the problem. This view depends on their experience. These experiences will form a cognitive system. When students encounter a mathematical problem, the knowledges are related to the problem can be brought up directly. Based on Farmaki and Paschos (2014), the idea that emerged after the process was called intuition. Intuition is defined as a cognition that appears subjectively by itself, immediately, erratically (Fischbein, 1999). In addition, according to Nickerson (2010) mathematical ideas essentially arise intuitively before being exposed to logical arguments. It shows that intuition has a fundamental role in determining students' mathematical reasoning abilities. Intuition is highly subjective and influenced by the level of one's cognitive development. Intuition has an indirect role in students' mathematical reasoning. Intuition is not explicitly visible in the process of providing an explanation of mathematical ideas. Therefore, intuition is an informal aspect of mathematical reasoning

2. Formal Aspect of Mathematical Reasoning

After students use their intuition to define a problem-solving strategy, they then attempt to solve the problem using mathematical concepts. Abstract ideas make students difficult to solve problems. Therefore, students need help to be able to communicate the concept. The symbols and notations in the question will be more easily understood if the students visually represent the ideas through diagrams or graphics. Representation of abstract ideas into visuals will help them to solve problems. Nickerson (2010) states that the power of representational systems as a vehicle of thought. Representation aids to reasoning and seeing relation on mathematical ideas. In addition to representation, provide a logical reason in every important statement in mathematical reasoning. Therefore, students' premises should have strong in power to making an argument. In addition, the ability in the inference method should also be well understood by the students when they have several premises. In some instances, often the premises that can be used only appear implicitly. Students must have the ability to locate the premise so that it can be used in proof. The basic concept of understanding of the relationship between the premises and the method of proof has an important role in mathematical reasoning.

B. Task Design

The examination of instruction and thinking processes was framed by concept of mathematical tasks, a close relative of academic tasks, a contruct that has been extensively employed to study the connection between teaching and leaning (Stein, et al, 1996). Mathematical tasks will give information about students cognitive development. Chapman (in Jonsson et al, 2016) appropriately designed mathematical tasks will (1) promote students' conceptual understanding of mathematics, (2) retain their interest in the task and (3) optimize their learning. Mathematical tasks are designed to have some features with certain purpose. These features will give information toward procedural and conceptual understanding of students.

Methodology

This study aims to describe the mathematical reasoning abilities of students reviewed based on informal and formal aspects. Intuition is an informal aspect that is described in this paper. Meanwhile, the formal aspect emphasizes the logical way of thinking in solving mathematical problems. We performed an analysis of the student's test results during the study of group theory to find out the descriptions of those aspects. Researchers compiled 12 items given in 3 different time periods. The informal, intuitive aspect can be described through the problem solving strategy. After that, the interview will provide a more detailed description of the informal aspects. Meanwhile, visual interpretation, connectivity in procedure and quality of explanation criteria are used to describe the formal aspects of students' mathematical reasoning.

Results and Discussion

We have prepared 12 task to describe students' mathematical reasoning. These task have 4 characteriscics i.e. the number of posible strategy to solve, visual interpretation, connectivity in procedure and requairement of explanation to each task. Detail of characterisctics task can be shown in table 1.

Table 1. Characteristics of Task

Characteristic of Task	Category	Number	Percentege
The number of stategy to solve the problem	Single	3	25
	Multiple	9	75
Visual interpretatiton	Posible	6	50
	impossible	6	50
Connectivity in procedure	No or few	2	17
	required	10	83
Explanation requaired	no or few	2	17
	Required	10	83

The table above shows that the characteristics of the task that are prepared more emphasized on the aspect of formal explanation by students. The tasks assigned to the students are more multi-strategy in its completion. It is intended to challenging in student cognitive activity. Thus we can detect about how student intuition. Problems that can be modeled visually and not arranged in equal in number. It aims to connect students' cognitive toward formal abstraction. Some problems are directed so that students can directly provide abstract ideas without making visualization in solving the problem. Most of the problems are structured to determine how students' mathematical reasoning abilities are. Any problems that require more explanation and interconnectedness in the procedure are the implicit premises that will be used to give a conclusion at every step.

The results show students' ability in mathematical reasoning is lacking. This is supported by some facts bellow.

1. Misconceptions in understanding the problem.

Understanding the problem is the first step in solving the problem. Misrepresentation of the problem will have a negative impact on clarification, justification and validation of mathematical ideas. Students try to connect some notation or symbol that are not related to the problem. One of students' problems is interpretation of group of integer modulo 3 $(\mathbb{Z}_3)$ and permutation group with 3 elements (S_3). The student was asked to find the operation of permutation grup, but he did for the operation on $\mathbb{Z}_3$. So, his argument and the conclusion are not matching.

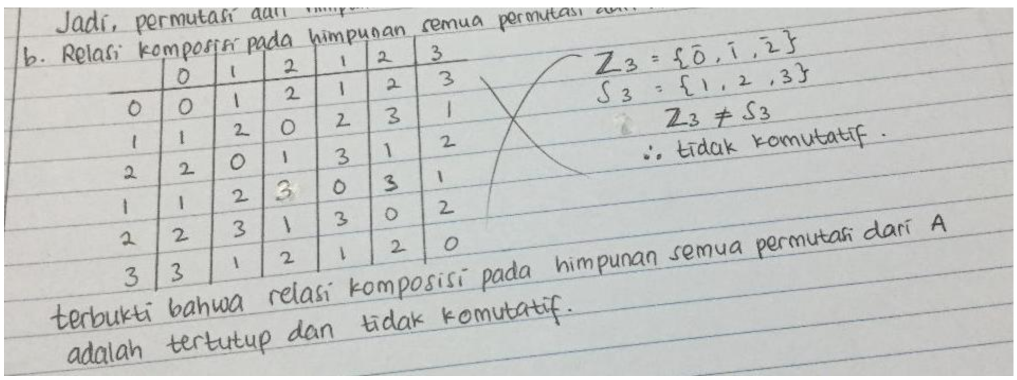

Figure 1. Students' answer about Permutation Group

2. Misconceptions in understanding the problem.

Mathematical reasoning skills rare depended on the strength of the premise used in the conclusion. Students have problems in determining which premises are involved to make a conclusion. As a result, students immediately write a conclusion without giving a logical reason. The student was asked to proved subgroup in two groups which are isomorphics. The argument of the student was very lack in premiss. He just wrote the proposition in the problem and then he arranged it to make an conclussion.

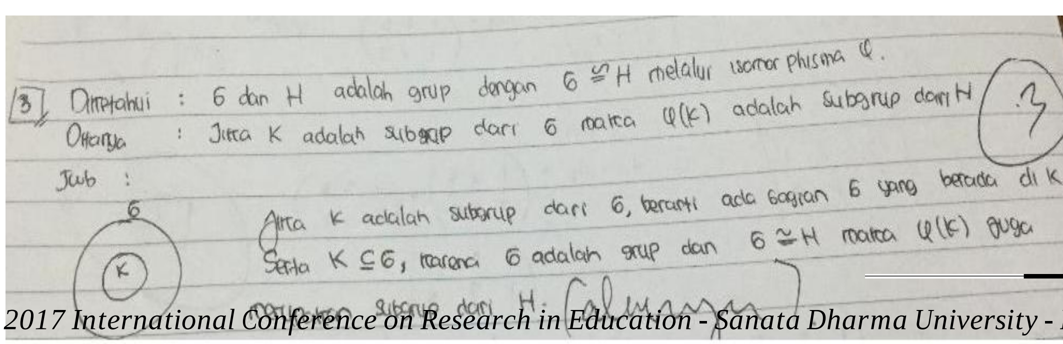

Figure 2. Students' Answer about Subgrup of Isomorphism Groups

3. Using incorrect argument in explaining.

   The problem involves the implicit premise also often make students mistaken in giving arguments. The premises are actually more aimed to determine the ability of students in relating mathematical ideas. Understanding the concept of clasics algebra is a prerequisite in order to make implicit premises become to explicit. There was a mistake to proved two groups are isomorphics about property of logarithm i.e. $\log(a+b) = \log a \cdot \log b$.

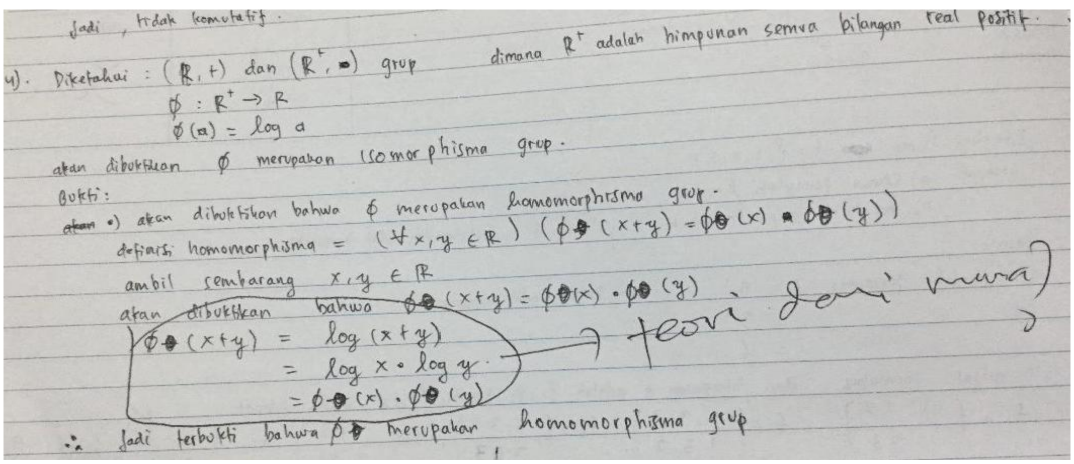

Figure 3. Student's Answer about Proofing Isomorphism in Groups

4. Sense of mathematical logic is still poor.

   Students' cognitive structure take an important rule in mathematical reasoning. Logic is a basic fondation in mathematical reasoning. The most basic problem observed by the students is how to give an argument for an implication. Students usually begin to prove the form of implications starting from the antecedents of the statement. In fact, students should use antecedents as a known assumption. This can be seen in the student proof in the following picture. Students were asked to proved that if K is subgroup of G and φ is isomorphism from group G to H then $\varphi(K)$ is subgroup of H. Students should start from the concequent of the propossition i.e. $\varphi(K)$ is subgroup of H. But, student did it from the antecedent. Logically, it is not true.

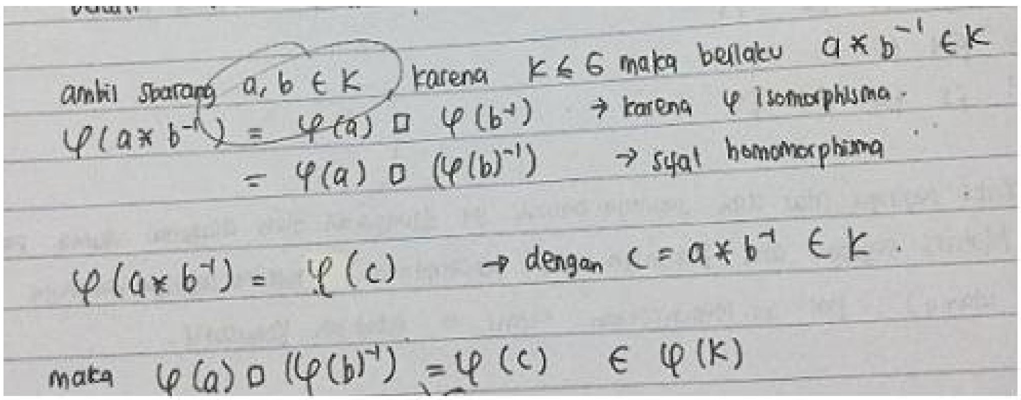

Figure 4. Students' Answer about Proofing Structure of Image of Isomorphism

Some results above indicate that the mathematical reasoning of the students is still poor. Formal aspects can be seen from the results of student answers to each given question. Characteristics for formal aspects that include connectivity and explanation characteristics have not adequately cleared by students.

Furthermore, a more in-depth investigation of the mathematical reasoning abilities of students. Interview aims to get more specific information on the characteristics of the problem, especially to get an overview of the informal aspects of the students' mathematical reasoning. Interview conducted on 3 students. The number of solving strategies and visual interpretation of the problem becomes the focus of the interview. The results obtained in the interview are given by the following table.

Table 2. Interviews' Result on Informal Aspect of Mathematical Reasoning

Student	Process of Answer the test	
	Multiple strategy	Visualize
A	3	1
B	0	1
C	4	2

Most students get stuck with the formal definition of some topics in group theory. Concequently, they always try to provide arguments with formal definitions they have understood. There are two conditions that arise based on this situation. First, the creativity of students in building an argument becomes impeded by focusing only on formal definitions that are very abstract. Second, the abstractions they use as supporting arguments tend to go wrong due to the inadequate logic of thinking in understanding the formal definition. The first impact is visible in determining possible strategies for solving a problem. Students focusing only on a formal definition tend to result students didn't think of alternative solutions to solve the problem. Of the nine questions that are expected to have many solutions to solve, the result is less than 50% of students are aware of it. The discussion bellow describe that students still have problems to get alternative strategis for the concept of generator elements in a group.

R : How do you find the generator of $\langle a^{21}\rangle \cap \langle a^{10}\rangle$ *for* a *is a element of group G and* $|a| = 24$*?*

S : First, I count all the elements of $\langle a^{21}\rangle$ *and* $\langle a^{10}\rangle$*. Then find the intersection ....*

R : Oh, its to long procedure. Do you have any alternative?

S : Hmm, I don't know. I think that is only one possible answer.

Based on the above interview, the students focused only on the formal definition of the generator and the intersection. They do not apply their experience during practice in doing on similar problems. The concept of LCM or GCD may can also be used to solve the problem.

Abstraction in group theory requires a great deal of learning experience over clasics algebra concepts. The experience will form a cognitive structure gradually. Learning experiences that have been recorded in the cognitive structure will help students determine alternative possibilities in solving problems. This is then referred to as intuition. Intuition is a necessary aspect of mathematical reasoning. Intuition is not directly related to the expected logical explanation in mathematical reasoning. Intuition appeared naturally, straightforward, self-evidence (Fischbein, 1999). Therefore, it can be said that intuition is an informal aspect for mathematical reasoning. This aspect will be very useful for students in building an abstract concept. Intuition will formulate the initial steps in resolving issues such as the use of strategy and problem interpretation. Students do not have good intuition yet. Students have not been able to restate learning experiences that are already structured in their cognitive structure. But intuition is very local. This means that student intuition in group theory may will be different from intuition in calculus. Therefore, it is natural that we state intuition is an informal aspect of mathematical reasoning.

Conclusion

Aspects of Students' mathematical reasoning are informal and formal aspect. Intuition is a informal aspect which are related to mathematical reasoning indirecly. Intuition takes role in choosing the alternative strategie to solve mathematical problems. Formal aspect takes role in justifying and verifying mathematcal concept. Explanation of arguments by using logical premises taking place at the heart of mathematical reasoning. Students' mathematical reasoning in exploring group theory still poor. At most student have not been able to connect between abstract concept of group with their experience in learning clasics algebra.

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