INTRODUCTION TO *MATLAB* FOR SOLVING AN ORDINARY DIFFERENTIAL EQUATION WITH INITIAL VALUE PROBLEM

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Abstract

One of the courses that existed in the mathematics education's lecture is the Numerical Method. The Numerical Method is a method of approach to finding approach solutions, and commonly used if an analytic problem problem is difficult to find. To introduce *MATLAB* to students, a learning activity plan based on problem-based learning method was developed. Stages in the learning activity plan include problem recognition, analytic solution search, numerical method introduction, manual numerical solution search, naumeris solution search using computer help, including use of *Microsoft Excel* as the introduction and *MATLAB*, and the meaning of the solution. The result of this planning is an example of ordinary differential equations with the initial value problem sought by the solution using the stages in the plan. The conclusion of this research are *MATLAB* is introduced to students because its use is more effective when compared to *Microsoft Excel* or manually ways, and *MATLAB* can be collaborated with problem based learning to create a new design of learning.

Keywords: *MATLAB*, ordinary differential equation, initial value problems, problem-based learning.

Introduction

Technology can be a powerful tool for transforming learning. It can help affirm and advance relationship between educators and students, reinvent our approaches to learning and collaboration, shrink long-standing equity and accessibility gaps, and adapt learning experiences to meet the needs of all learners (U.S. Department of Education, 2017). There are so many product of technology that has been created with the purpose to make a better experiences in learning. One of these product is *MATLAB*. There are some research that explain about *MATLAB*, one of them is in a research from GAO Feng (2011) about application of *MATLAB* in mathematical analysis, that presents several cases of *MATLAB* applications in mathematical analysis and the conclusion of this research is *MATLAB* can be a significant tool in mathematical analysis. One of the subjects taught to students of mathematics education at bachelor level is numerical method. Learning in the numerical course teaches the students to solve a case using a variety of ways. Mathematical problems can be solved by numerical methods such as systems of linear equations, systems of differential equations, and other systems of equations. Not all mathematical problems can be easily resolved analytically. Numerical methods are used to find solutions to a problem, especially one that is difficult to solve using analytical methods. Numerical calculations can be done manually or with the help of computers. Numerical calculations often require precision so that computer programming help such as *Microsoft Excel* or *MATLAB* is required. *Microsoft Excel* is a fairly simple and easy-to-use application for calculation, but the use of *Microsoft Excel* become less effective if iteration is too much, not to mention if iterative schema is complicated. One other application that can help numerical calculation is *MATLAB*. In this paper we want to answer about the question why using *MATLAB* to solved the problem and we also want to know if *MATLAB* can be collaborated with problem based learning to make a new design of learning.

Theory

A. Ordinary Differential Equations with Initial Value

Ordinary differential equations (ODE) is an aquation for a missing function (or function) in term of derivatives of those functions. Recall that the detivative of a function $y(t)$ is denoted either by $y'(t)$ or by dy/dt. Then the derivative $y'(t_0)$ gives the slope tangent line to the graph of the function $y(t)$ at the point (t_0, y_0) (Devaney, 2011). The best-known differential equation (and essentially the first example of a differential equation) is Newton's second law of motion. Drop an object from the rooftop. If y measuresthe position of the center of mass of the object, then we would like to know its position at time t, that is, $y(t)$. Newton's law tells us that mass times acceleration is equal to the force on the object. So, if m is the mass, then we have $my" = F(y)$, where F is the force acting on the object when it is in position $y(t)$. So we have a differential equation for $y(t)$ (Devaney, 2011).

A differential equation with additional terms to the unknown function and its derivatives, all given to the same value for the free variables, is an initial value problem (Nugraha, 2011). For example, we look at the unlimited population growth model from biology. Suppose we have a species living in isolation (with no predators, no overcrowding, and no emigration) and we want to predict its population as a function of time. Call the population $y(t)$. Our assumption is that the rate of growth of the population is directly proportional to the current population. This translates to ODE $y' = ky$. Here k is constant (a parameter) that depends on which species we are considering. We wish to find the solution of an initial value problem, that is, a specific solution of the ODE that satisfies $y(0) = y_0$ where y_0 is given initial population.

B. *MATLAB*

MATLAB is an interactive system for doing numerical computations. *MATLAB* relieves we of a lot of the mundane tasks associated with solving problems numerically. This allows you to spend more time thinking, and encourages we to experiment. *MATLAB* makes use of highly respected algorithms and hence we can be condent about our results. Powerful operations can be performed using just one or two commands. We can build up our own set of functions for a particular application (Griffiths, 2015). *MATLAB* stands for *MATRIX LABORATARY* (Matrix Laborotory), which was first used by Dr. CleveMoler in New Mexico University United States to teach courses of liear algebra. The basic data unit is a matrix without dimension restriction. In *MATLAB*, computing become extremely easy. Dr. Moler in 1984 launched the official version of the software, in the later editions, he also gradually added to the control system, system identification, signal processing and communications, more than a toolbox, so *MATLAB* became to be widely used in automical control, image signal processing, biomedical engineering, signal analysis, optimization and other fields (Feng, 2011).

In pure mathematics, since *MATLAB* is an integrated comuter software which has three function: symbolic computing, numerical computing and graphic drawing, *MATLAB* is capable to carry out many functions including computing polynomials and rational polynomials, solving equations and computing many kind of of mathematical expression. One can also used *MATLAB* to calculate the limit, derivative, integral and Taylor Series of some mathematical expressions. With *MATLAB*, the graph of functions with one or two variables can be easily drawn in selected domain (Feng, 2011).

C. Problem Based Learning

Problem-based learning has often been understood simply as a method of learning. Correspondingly, many kinds of pragmatically based pedagogical applications and development project are described as PBL. Problem-based learning has also been investigated within the context of education, altough the theoretical basis of problem-based learning is closely connected with learning at work (Poikela, 1998; Karila & Nummenmaa, 2001; Poikela & Jarvinen, 2001; Poikela & Poikela, 2001 in Poikela & Nummenma, 2006). Problem-based learning as an educational methodology started at the McMaster University, Canada in 1969. Problem-based on actual clinical cases were used as focl points in a medical program (Ee, 2009). The problem-based learning process essentially consists of the following stage: (1) meeting the problem; (2) problem analysis and generation of learning issues; (3) discovery and reporting; (4) Solution presentation and reflection; and (5) overview, integration, and evaluation, with self-directed learning bridging one stage and the next (Tan, 2003 in Tan, 2009).

Methods (Design for The Learning Activity)

Broadly speaking, the activity design composed includes 6 stages, which are:

1. Problem Introduction

   The problem recognition process aims to provide a context of problems to students, so that their purpose of learning a method becomes clearer. In the process of introducing this problem, students are expected to know what is really a problem in a case. At this stage students analyze the problem, then determine a hypothesis related problems.

2. Search for Analytic Solutions

   This stage is the stage where students are invited to find a suitable strategy to solve the problem analytically. Students are then expected to apply the strategies they have to find an analytical solution.

3. Introduction of the Approach Methods

   The next stage is the introduction of approach methods. At this stage students are introduced to alternative methods that can also be used to find solutions to the problems. Students then analyze the methods until finally making a plan to find a solution of the problem. The draft strategy is generally an iterative scheme.

4. Search for The Approach Solutions Manually

In the next stage students are expected to apply strategies they have designed manually. The goal is to make the students really understand the iterative schemes they have previously obtained.

5. Search for The Solutions Using Computer Help

This stage involves searching the approach solutions using *Microsoft Excel* and *MATLAB* applications. In using the approach method, generally iteration required quite a lot of number. When the required iteration and iterative scheme used are quite complicated, then the process of finding solutions will take quite a long time and requires considerable accuracy. Therefore, in order for the process of finding solutions to work more effectively, computer assistance is required.

First, students are excpected to use Microsoft Excel to find an approach solution, by applying iterative schema into the application. To compare, an analytic solution scheme (exact solution) is provided, so we can look for errors at each point (the corresponding per-cell error).

The next stage, students are introduced to MATLAB, because in some cases, often the use of Microsoft Excel is considered less effective. To use MATLAB students must have already understood the basic algorithms in programming. Since this process involves an iterative schema, the algorithm that must be mastered is a repetition algorithm.

6. The Meaning of Solutions

In the last stage of the learning activities, students are excpected to review the problems that have been previously given. Students are then asked to link the solutions that have been obtained with the context of the problem, to see whether the results match or not with the hypothesis that has been proposed at the beginning of the learning process.

Result and Discussion (An Example of ODE with Initial Value Problem)

A. Problem Introduction

An object that made from metal, 0.5 kg heavy and the initial temperature is 300^0(573 K) dipped in water that have temperature at 25^0(298 K), where there is a natural conventionally cooling process occurs.

$$\frac{dT}{dt} = \frac{A}{\rho cv} h_c(298 - T) \tag{1}$$

In the equation (1), the temperature is in K (Kelvin) unit, with the constants:

density, $\rho = 300$ kg/m^3

volume , v = 0,005 m^3

surface area, A= 0,25 m^2

Heat, c = 900 J/KgK

Heat transfer coefficient, h_c = 300W/m^{2}K

From the problem, we will look for the temperature plot of the object for 5 minutes (quoted from Kosasih, 2006).

Hypothesis: Logically, when a metal cools, then the plot temperature for 5 minutes will decrease from the initial temperature to a certain temperature.

B. Analytic Solutions

From the given problem, then we can obtain the analytic solution from that problem like this:

From the given problem we obtaining a system for an ordinary differential equation (ODE) with initial value problem:

$$\begin{cases} \frac{dT}{dt} = \frac{A}{\rho cv} h_c(298 - T) & (2) \\ T(0) = 573 & (3) \end{cases}$$

For $0 \le t \le 5$.

Where:

t = time (in minute)

T = temperature

Analitical solution for the equation system (2) and (3) as follows:

$$\frac{dT}{(298-T)} = \frac{A}{\rho cv} h_c dt \quad (4.a)$$

$$\int \frac{dT}{(298-T)} = \frac{A}{\rho cv} h_c \int dt \quad (4.b)$$

$$-\ln |(T-298)| = \frac{A}{\rho cv} h_c t + C \quad (4.c)$$

$$T = 298 + e^{-\frac{A}{\rho cv} h_c t} . e^c \quad (4.d)$$

From equation (3) was known that $T(0) = 573$, then e^c can be found by this information:

$$573 = 298 + e^{-\frac{A}{\rho cv} h_c \cdot 0} . e^c \quad (5.a)$$

So that: $$e^c = 275 \quad (5.b)$$

By substituting (5.b) to (4. d) we get the analytic solution:

$$T = 298 + 275 e^{-\frac{A}{\rho cv} h_c t} \quad (6)$$

C. Introduction of the Approach Methods

At this stage, using the problem in equation (1), two approach methods are considered suitable to find the solution, that are Euler Method and Heun Method. Students are invited to actively communicate the two methods. One of the alternatives for the introduction of these two methods of approach is counselors with some discussion with the students describe the Taylor Series to get an iterative scheme of Euler Method for the solution completion. After that students are asked to discuss the iterative scheme obtained for Heun Methods based on the information they have gained about the Euler Method and the idea of Method Heun. Here is one form of Euler Method description and Heun Method (Aminuddin, 2006).

1. Euler Method

   Euler Method is a method for solving the ODE using the Taylor series.

$$y' = \frac{dy}{dx} = f(x,y) \quad (7)$$

   The iterative scheme for Euler Method is :

$$y_{i+1} = y_i + \Delta x f(x,y) \quad (8)$$

   From the equation (8) can be concluded that the scheme of the solution for the previous problem, using the Euler Method is:

$$T_{(i+1)} = T_i + hf(t_i, T_i) \quad (9)$$

   Note that $h = \Delta t$, for $i = 0, 1, 2, 3, \ldots$

2. Heun Method

   This method estimates two gradients at intervals, that is at the beginning and end points. The best value is obtained from averaging between the gradient at the beginning and end points. The gradient at the earliest end of the interval obtained by the Euler Method is expressed in the form of equation (7). Then linear extrapolation is applied to the value of y_{i+1}.

$$y^*_{i+1} = y_i + \Delta x f(x,y) \quad (10)$$

$$y_{i+1} = y_i + \frac{1}{2}\Delta x(f(x_i, y_i) + f(x_{i+1,} y^*_{i+1}) \quad (13)$$

   From the equation (10) and (13), obtained the scheme of resolving the initial value problem that already described above using Heun method, that is:

$$T^*_{i+1} = T_i + hf(t_i, T_i) \quad (11.a)$$

$$T_{(i+1)} = T_i + \frac{1}{2}h[f(t_i, T_i) + f(t_{i+1}, T^*_{i+1})] \quad (11.b)$$

Note that $h = \Delta t$, for $i = 0, 1, 2, 3 \ldots$

D. The Approach Solutions Manually

Consider $h = 1$, which mean that the interval of time for predicting the temperature every minutes. So that, in the 5 minutes, we need 5 numbers of iteration to get the temperature at the 1st, 2nd, 3th, 4th, and 5th minute (in this stage we are not solve the all of the problem yet, because the problem ask the plot for 5 minutes). Here is one form of solution using the Euler Method and The Heun Method.

1. Using Euler Method

Using the iterative scheme (9), we get the solutions for $h = 1$ are:

$T_i = T_0 + hf(t_0, T_0)$

$T_1 = 573 + \frac{3500 \times 0.25}{900 \times 0.005} \times (298 - 573) = 394.7593$ (The object temperature at the first minute is $394.7593\ K$)

$T_2 = T_1 + hf(t_1, T_1)$

$T_2 = 557{,}72 + \frac{3500 \times 0.25}{900 \times 0.005 \times 300} \times (298 - 394.7593) = 332.0449$ (The object temperature at the second minute is $332.0449\ K$)

The iteration process is done until the 5th minute.

2. Using Heun Method

Using the iterative schemes (11.a) and (11.b), we obtained the solutions for $h = 1$, which are:

$T_1^* = T_0 + hf(t_0, T_0)$

$T_1^* = 573 + \frac{3500\times0.25}{900\times0.005\times300} \times (298 - 573) = 394.7593$

$T_1 = T_0 + \frac{1}{2}[f(t_0, T_0) + f(t_1, T_1^*)]$

$T_1 = 573 + \frac{1}{2} \times \frac{3500\times0.25}{900\times0.005\times300}[(298 - 573) + (298 - 394.7593)] = 452.5225$ (the object temperture at first minute is $452.5225\ K$)

$T_2^* = T_1 + hf(t_1, T_1)$

$T_2^* = 558{,}15 + \frac{3500\times0{,}25}{900\times0{,}005\times300} \times (298 - 452.5225\ K) = 352.3690$

$T_2 = T_1 + \frac{1}{2}h[f(t_1, T_1) + f(t_2, T_2^*)]$

$T_2 = 543{,}697 + \frac{1}{2} \times \frac{3500\times0{,}25}{900\times0{,}005\times300}[(298 - 558{,}15) + (298 - 352.3690)] = 384.8262$

(the object temperture at second minute is $384.8262\ K$).

The iteration process is done until the 5th minute.

E. The Approach Solutions Using Computer Help

There is some calculation results using *Microsoft Excel* for Euler Method and Heun Method, and the comparison of error from both of that method.

Table. 1. Calculation results for $h = 1$

N	t(i)	Euler	Analytic	Error	t(i)	T*	Heun	Analytic	Error
0	0	573	573	0	0		573	573	0
1	1	394,7593	441,8287	47,0694	1	394,7593	452,5225	441,8287	10,6938
2	2	332,0449	373,2243	41,1794	2	352,3690	384,8262	373,2243	11,6018
3	3	309,9788	337,3433	27,3646	3	328,5499	346,7876	337,3433	9,4443
4	4	302,2148	318,5771	16,3623	4	315,1660	325,4137	318,5771	6,8367
5	5	299,4830	308,7621	9,2791	5	307,6456	313,4038	308,7621	4,6417

Table. 2. Calculation results for $h = 0.25$

N	t(i)	Euler	Analytic	Error	t(i)	T*	Heun	Analytic	Error
0	0	573	573	0	0		573	573	0
1	0,2	528,439	531,862	3,422	0,2	528,439	532,050	531,862	0,187

	5	8	7	9	5	8	0	7	3
2	0,5	491,1000	496,8791	5,7791	0,5	494,1252	497,1979	496,8791	0,3188
3	0,75	459,8107	467,1287	7,3180	0,75	464,9204	467,5355	467,1287	0,4068
4	1	433,5914	441,8287	8,2373	1	440,0645	442,2901	441,8287	0,4614
5	1,25	411,6205	420,3133	8,6928	1,25	418,9098	420,8040	420,3133	0,4907
6	1,5	393,2098	402,0164	8,8066	1,5	400,9052	402,5174	402,0164	0,5010
7	1,75	377,7823	386,4566	8,6743	1,75	385,5817	386,9538	386,4566	0,4972
8	2	364,8546	373,2243	8,3697	2	372,5400	373,7078	373,2243	0,4834
9	2,25	354,0217	361,9715	7,9498	2,25	361,4403	362,4342	361,9715	0,4627
10	2,5	344,9441	352,4020	7,4579	2,5	351,9935	352,8394	352,4020	0,4374
11	2,75	337,3374	344,2640	6,9266	2,75	343,9534	344,6733	344,2640	0,4093
12	3	330,9633	337,3433	6,3800	3	337,1105	337,7232	337,3433	0,3799
13	3,25	325,6220	331,4580	5,8359	3,25	331,2866	331,8081	331,4580	0,3501
14	3,5	321,1462	326,4530	5,3067	3,5	326,3299	326,7737	326,4530	0,3208
15	3,75	317,3957	322,1967	4,8010	3,75	322,1113	322,4891	322,1967	0,2924
16	4	314,2529	318,5771	4,3242	4	318,5209	318,8424	318,5771	0,2653
17	4,25	311,6193	315,4990	3,8797	4,25	315,4652	315,7388	315,4990	0,2398
18	4,5	309,4125	312,8813	3,4688	4,5	312,8645	313,0973	312,8813	0,2161
19	4,75	307,5632	310,6552	3,0920	4,75	310,6510	310,8492	310,6552	0,1940
20	5	306,0136	308,7621	2,7485	5	308,7672	308,9358	308,7621	0,1737

From both tables above obtained that the approach solution using Euler Method and Heun Method for $h = 1$ and $h = 0.25$. From that tables we can see that for the smaller value of h then the approach solutions become morea ccurate to the analytic solutions. The temperatures plot for the metal object can be created using *Microsoft Excel* or using *MATLAB* (in this case we will using the *MATLAB's* programs to make the temperatures plot for the problem)

This is one of the examples for the programs which can be made by the students using *MATLAB* application.

1. *MATLAB's* program for the function

```
function z=fk(t,T)  % function on the right hand of ODE
   z=0.25*3500*(298-T)/(300*900*0.005);    % Equation
end
```

2. *MATLAB's* program for the Euler Method

```
clear all;
clc
a=0;                      % initial time
b=5;                      % the end time (in minutes)
h=1/60                      % time interval (delta t)
t=a:h:b;
N=length(t);                % the number of points t
for i=2:N
   t(1)=0;                % initial value of t
   T(1)=573;              % initial value of T
   T(i)=T(i-1)+h*fk(t(i-1),T(i-1)); % Euler Method scheme
end
TE= 298+275*exp(-((0.25*3500)/(300*900*0.005))*t); % Analytic T
plot(t,TE,'k', t,T,'b-')            % solution graph
legend('TEksak','TNumeris')
xlabel('t')
ylabel('T')
Error=(1/N)*sum(abs(TE-T))  % error calculation
tabel =[T' TE'];           % making a table
disp(tabel)                % display the table
```

3. *MATLAB's* Program for Heun Method

```
clear all;
clc
a=0;                      % initial time
b=5;                      % the end time (in minute)
h=1/60                      % time interval (delta t)
t=a:h:b;
```

```
N=length(t);              % the number of points t
for i=2:N
   t(1)=0;              % initial value of t
   T(1)=573;              % initial value of T
   % Scheme of Heun Method
   Tib=T(i-1)+h*fk(t(i-1),T(i-1)); % T(i) star
   T(i)=T(i-1)+0.5*h*[fk(t(i-1),T(i-1))+fk(t(i),Tib)];
end
TE= 298+275*exp(-((0.25*3500)/(300*900*0.005))*t); % Analytic T
plot(t,TE,'k', t,T,'b-')             % Solution graph
legend('TEksak','TNumeris')
xlabel('t')
ylabel('T')
Error=(1/N)*sum(abs(TE-T))  % error calculation
tabel =[T' TE'];            % making a table
disp(tabel)               % display the table
```

We can change the function depends on the problem. Other than that, in the program for Euler and Heun Methods, the a, b and h variables can be customized with the given problem.

F. The Meaning of Solutions

1. Euler Method

Using the example of the program in the previous stage, we obtain the solution graph like this.

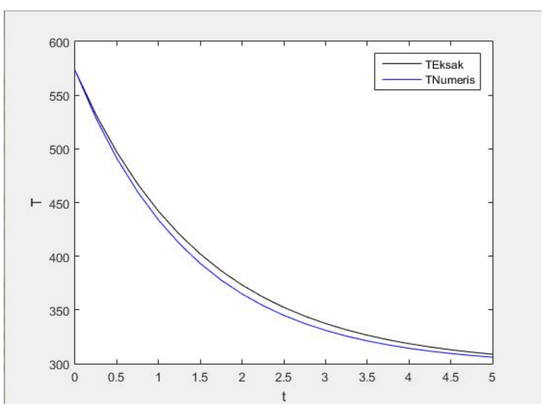

Figure 1. Euler's solution graph for $h = 0.25$

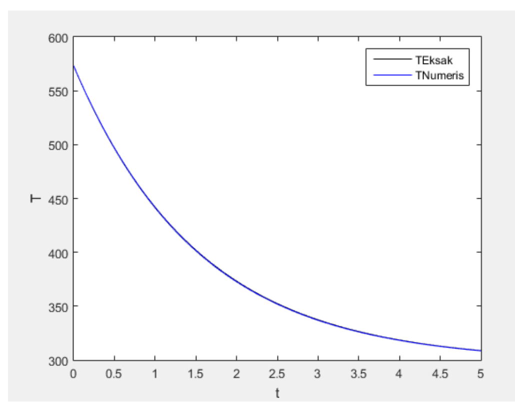

Figure 2. Euler's solution graph for $h = \frac{1}{60}$

By looking at Fig. 1. and Fig. 2. we can see that a smaller h value will give us more accurate approximation solutions (using the program obtained the total error for $h = 0.25$ is 5.7844 while the total error for $h = \frac{1}{60}$ is 0.3825)

2. Heun Method

Using the example of the program in the previous stage, we obtain the solution graph like this.

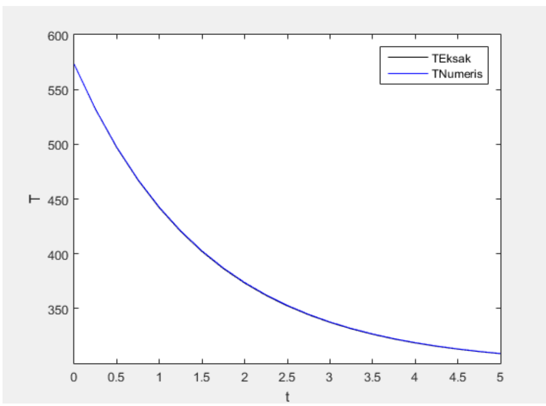

Figure 3. Heun's solution graph for $h = 0.25$

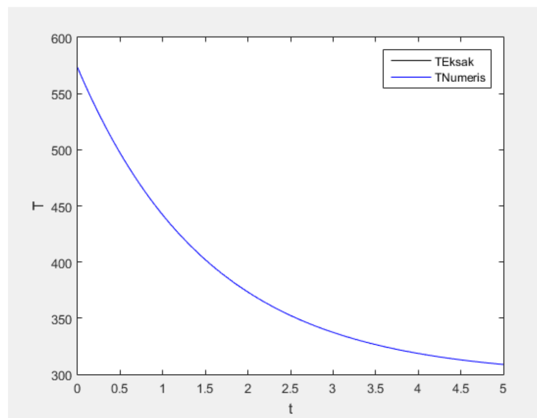

Figure 4. Heun's solution graph for $h = \frac{1}{60}$

From Fig. 3. and Fig. 4. we can see that the two approach graphs are quite accurate when compared to the previous method for each of the same h values (from the program the total error is 0.3375 for $h = 0.25$, while the total error for $h = \frac{1}{60}$ is 0.0014). Overall, we can see from the solution graph (both in Euler Method and Heun Method) that the metal temperature decreased from the initial temperature of 573 K to close to 300 K within 5 minutes (from the temperature calculation after 5 min about 308 K). In addition we can see that in this case, the use of Method Heun is more effective than the use of Euler Method, since the error obtained from Method Heun is less than the error obtained from the Euler Method, for the same time interval.

Conclusion

MATLAB is introduced to students because its use is more effective when compared to manual or using *Microsoft Excel*, especially when iterations are needed in large numbers and iterative schemes are quite complicated. The use of *MATLAB* can minimize errors caused by inaccuracy in calculations. To use *MATLAB* students are required to master basic programming algorithms, especially repetition algorithms, and fully understand the iterative scheme of the approach method to be used. Also, we can see that *MATLAB* can be collaborated with problem based learning to create a new method of learning.

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