# Boundedness of Littlewood-Paley-Stein (LPS) Operator in Lebesgue Space with an Improved Sufficient Condition

**Pebrudal Zanu[1], Wono Setya Budhi[2], Yudi Soeharyadi[3]**

Reaserch Division of Analysis and Geometry, Mathematics and Natural Science Faculty, Institut Teknologi Bandung

E-mail: [1]pebrudal@gmail.com, [2]wono@math.itb.ac.id, [3]yudish@math.itb.ac.id

**Abstract.** Littlewood-Paley-Stein (LPS) operator is operator that maps a function to square function associated with a function $\psi$, that is $f \mapsto \left(\sum_{j\in\mathbb{Z}} |\psi_{2^{-j}} * f|^2\right)^{1/2}$. Littlewood-Paley established a sufficient condition for boundedness of LPS operator in classical Lebesgue Space. The condition is expressed in term of bound for sum $|\psi| + |\nabla\psi|$. In this article, we investigate and prove boundednesss of LPS operator with a generalized version for the bound from the original version.
**Keywords**: Boundedness of Operator, LPS Operator, Lebesgue Space

## 1. Introduction

In this work we will prove the boundedness of Littlewood-Paley operator type square function in Lebesgue space with an improved sufficient condition. This operator has introduced by Stein is called Littlewood-Paley-Stein (LPS) operator [2].

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ is be an integrable function and $\psi \in \mathcal{C}^1(\mathbb{R}^n)$ with condition

(i). $\widehat{\psi}(0) = 0$, where $\widehat{\psi}$ is Fourier Transform of $\psi$.
(ii). There exists $B > 0$ such that for every $x \in \mathbb{R}^n$ then $|\psi(x)| + |\nabla\psi(x)| \leq \frac{B}{(1+|x|)^{n+1}}$.

Define LPS operator $g_\psi$ by

$$g_\psi(f) := \left(\sum_{j\in\mathbb{Z}} |\Delta_j(f)|^2\right)^{\frac{1}{2}},$$

where $\Delta_j(f) := \psi_{2^{-j}} * f$ for every $f$ is measurable function, $\psi_{2^{-j}}(x) := 2^{jn}\psi(2^j x)$ and $j \in \mathbb{Z}$.

Let $1 \leq p < \infty$, we define weak Lebesgue space $wL^p(\mathbb{R}^n)$ by the set of all measurable function in $\mathbb{R}^n$ such that

$$\|f\|_{wL^p} = \sup_{t>0} t|\{x \in \mathbb{R}^n : |f(x)| > t\}|^{\frac{1}{p}} < \infty$$

Littlewood-Paley showed the following boundedness in [1]

**Theorem 1.** *Let $1 < p < \infty$. If $\psi$ integrable function in $\mathbb{R}^n$ , $\psi \in \mathcal{C}^1(\mathbb{R}^n)$, satisfies condition (i) and (ii) then $g_\psi$ is*

*(1) bounded from $L^1(\mathbb{R}^n)$ to $wL^1(\mathbb{R}^n)$.*

*(2) bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Hereafter, we say bounded on $L^p(\mathbb{R}^n)$.*

The aim of this article to show the boundedness of $g_\psi$ from $L^1(\mathbb{R}^n)$ to $wL^1(\mathbb{R}^n)$ and the boundedness on $L^p(\mathbb{R}^n)$ with with weaker condition for (ii) become

$$|\psi(x)| + |\nabla\psi(x)| \leq \frac{B}{(1+|x|)^{n+\alpha}}, \quad \alpha > 0 \tag{1}$$

Our strategy to prove boundedness from $L^1(\mathbb{R}^n)$ to $wL^1(\mathbb{R}^n)$ by show $g_\psi$ is Calderón-Zygmund operator, while the boundedness on $L^p(\mathbb{R}^n)$ using Marcinkiewicz interpolation Theorem and duality Theorem. Marcinkiewicz interpolation suffice to show $g_\psi$ is bounded from $L^1(\mathbb{R}^n)$ to $wL^1(\mathbb{R}^n)$ and bounded on $L^p(\mathbb{R}^n)$.

Note that for $\alpha \geq 1$ the boundedness easy to see by Theorem 1. So, we suffice to proof the boundedness for $0 < \alpha < 1$.

The following is definition of Calderón-Zygmund kernel and Calderón-Zygmund operators

**Definition 2** (Modification, [1], p.402)**.** *Let $\vec{K} : \mathbb{C} \to \ell^2$ be a bounded sublinier operator from $\mathbb{C}$ to $\ell^2$. $\vec{K}$ is Calderón-Zygmund kernel if only if satisfying following this condition*

*(K1). There exists $\vec{K_0}$ bounded linier operator such that*

$$\lim_{\varepsilon\to 0} \left\| \int_{\varepsilon \leq |y| \leq 1} \vec{K}(y)dy - \vec{K_0} \right\|_{\ell^2} = 0$$

*(K2). There exists $C_n > 0$ such that for every $x \in \mathbb{R}^n \backslash \{0\}$*

$$\|\vec{K}(x)\|_{\ell^2} \leq C_n|x|^{-n}$$

*(K3). For every $y \in \mathbb{R}^n \backslash \{0\}$ satisfies*

$$\int_{|x|\geq 2|y|} \|\vec{K}(x-y) - \vec{K}(x)\|_{\ell^2} dx \leq C_n.$$

*Calderón-Zygmund operator $\|\vec{T}\|_{\ell_2}$ associated with kernel $\vec{K}$ as follows.*

$$\vec{T}f(x) = \lim_{\varepsilon\to 0} \int_{|y|\geq\varepsilon} \vec{K}(y)f(x-y)dy. \tag{2}$$

**Proposition 3.** *Suppose $\psi$ integrable function in $\mathbb{R}^n$, $\psi \in \mathcal{C}^1(\mathbb{R}^n)$ and $\psi$ satisfy the following condition*

(C1). $\widehat{\psi}(0) = 0$
(C2). Satisfy (1), where $0 < \alpha < 1$.

Let $\vec{K}(x) := \{\psi_{2^{-j}}(x)\}_{j=-\infty}^{\infty}$. For $f \in C_0^\infty(\mathbb{R}^n)$ and $\vec{T}(f) := \vec{K} * f$ then $\vec{K}$ is Calderón-Zygmund kernel and $g_\psi = \|T(\cdot)\|_{\ell^2}$ is a Calderón-Zygmund operators.

## 2. Proof of Proposition 3 and the Aplication

For $\vec{K}$ is bounded linier operator and existence of $\vec{K}_0$ proved in [1] on Theorem 6.1.2, and the proof does not depend on the new condition. So, we suffice to prove for condition (K2) and (K3).

(K2). For $0 < \alpha < 1$, let $x \in \mathbb{R}^n$, and choose $k \in \mathbb{Z}$ such that $2^{(k-1)/\alpha} \le |x| \le 2^{k/\alpha}$

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \frac{2^{2jn}|x|^{2n}}{(1+|2^j x|)^{2n+2\alpha}} &\le \sum_{j\le -k} 2^{2jn}|x|^{2n} + \sum_{j>-k} \frac{1}{2^{2j}|x|^{2\alpha}} \\
&\le \sum_{j\le -k} 2^{(2jn)/\alpha}|x|^{2n} + \sum_{j>-k} \frac{1}{2^{2j}|x|^{2\alpha}} \\
&\le \sum_{j\le -k} 2^{\frac{2(j+k)n}{\alpha}} + \sum_{j>-k} \frac{1}{2^{2(j+k-1)}} \\
&\le \sum_{t=0}^{\infty} \frac{1}{2^{\frac{2tn}{\alpha}}} + \sum_{t=0}^{\infty} \frac{1}{2^{2t}} = C_{n,\alpha}^2.
\end{aligned}$$

(K3). For $0 < \alpha < 1$, by using mean value theorem, there exists $\theta \in (0,1)$ such that for $|x| \ge 2|y|$ satisfy $|x - \theta y| \ge \frac{1}{2}|x|$, and

$$|\psi_{2^{-j}}(x-y) - \psi_{2^{-j}}(x)| \le 2^{j(n+1)}|\nabla\psi(2^j(x-\theta y))| \le B2^{nj}(1+2^{j-1}|x|)^{-(n+\alpha)}2^j|y|. \tag{3}$$

By using triangle inequality,

$$|\psi_{2^{-j}}(x-y) - \psi_{2^{-j}}(x)| \le 2B2^{nj}(1+2^{j-1}|x|)^{-(n+\alpha)}. \tag{4}$$

From (3), (4) and for $\gamma \in [0,1]$ is satisfied,

$$|\psi_{2^{-j}}(x-y) - \psi_{2^{-j}}(x)| \le 2^{1-\gamma}B2^{nj}(2^j|y|)^\gamma(1+2^{j-1}|x|)^{-(n+\alpha)}. \tag{5}$$

Choose $\gamma = \alpha$ for $2^j < \frac{2}{|x|}$, and $\gamma = \frac{\alpha}{2}$ for $2^j \ge \frac{2}{|x|}$, from (5) we obtain

$$\|\vec{K}(x-y) - \vec{K}(x)\|_{\ell^2} \le K_n B\left(|y|^\alpha|x|^{-n-\alpha} + |y|^{\frac{\alpha}{2}}|x|^{-n-\frac{\alpha}{2}}\right).$$

By using radial integral for $\alpha > 0$ there exists $C_n > 0$ such that

$$\int_{|x|\ge 2|y|} \|\vec{K}(x-y) - \vec{K}(x)\|_{\ell^2} dx \le C_n B.$$

This step is completed for our prove to $\vec{K}$ is Calderón-Zygmund kernel. $\vec{T}$ associated with $\vec{K}$ in relation 2, it implies $g_\psi = \|\vec{T}(\cdot)\|_{\ell^2}$ is Calderón-Zygmund Operators.

From Thoma [6] the Calderón-Zygmund operator implies the it is well-defined and bounded from $L^1(\mathbb{R}^n)$ to $wL^1(\mathbb{R}^n)$.

## 3. Operator $g_\psi$ is bounded on $L^2(\mathbb{R}^n)$

We will give a new additional condition to ensure that the boundedness on $L^2(\mathbb{R}^n)$ holds. The new condition is not necessary condition and we still have replace this condition with other one.

**Proposition 4.** *Let $\psi$ is function satisfies condition in Propotition 3 and following condition*

*C3. For any $y \in \mathbb{R}^n \backslash \{0\}$ and $0 < \alpha < 1$ satisfy*

$$\int_{\mathbb{R}^n} |\psi(x) - \psi(x-y)| dx \leq B|y|^\alpha,$$

*then $g_\psi$ is bounded on $L^2(\mathbb{R}^n)$.*

*Proof.* For $0 < \alpha < 1$. Note that $|e^{-2\pi i x\cdot\xi} - 1| \leq |\xi|^{\frac{\alpha}{2}}|x|^{\frac{\alpha}{2}}$, then by (C1) and (C2) we obtain

$$\left|\widehat{\psi}(\xi)\right| \leq \int_{\mathbb{R}^n} |e^{-2\pi i x\cdot\xi} - 1||\psi(x)| dx \leq |\xi|^{\frac{\alpha}{2}} \int_{\mathbb{R}^n} |x|^{\frac{\alpha}{2}}|\psi(x)| dx \leq C_\alpha B|\xi|^{\frac{\alpha}{2}}. \tag{6}$$

Note that

$$\widehat{\psi}(\xi) = -\int_{\mathbb{R}^n} \psi(x-y) e^{-2\pi i x \xi} dx, \quad \text{with} \quad y = \frac{\xi}{2|\xi|^2},$$

such that from (C3) condition we obtain

$$|\widehat{\psi}(\xi)| = \left|\frac{1}{2}\int_{\mathbb{R}^n} \psi(x) - \psi(x-y) e^{-2\pi i x \xi} dx\right| \leq B|y|^\alpha = \left(\frac{1}{2}\right)^{\alpha+1} B|\xi|^{-\alpha}. \tag{7}$$

From (6) and (7), we obtain $\eta$, such that $|\eta| = |\xi|^\alpha$, and

$$|\widehat{\psi}(\eta)| \leq C_\alpha B\sqrt{\eta} \quad \text{and} \quad |\widehat{\psi}(\eta)| \leq C_\alpha B|\eta|^{-1}. \tag{8}$$

Furthermore, take $\eta_0 \neq 0$, the least $k \in \mathbb{Z}$ such that $|2^{-k}\eta_0| \leq 1$. If $j \geq k$ then $|2^{-j}\eta_0| \leq 2^{k-j}$ and if $j \leq k$ then $|2^{-j}\eta_0| > 2^{k-j}$. We obtain

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^{-j}\eta_0)|^2 = \sum_{j\geq k} |\widehat{\psi}(2^{-j}\eta_0)|^2 + \sum_{j<k} |\widehat{\psi}(2^{-j}\eta_0)|^2 \leq C_\alpha^2 B^2. \tag{9}$$

By using (9) and Plancherel theorem,

$$\begin{aligned}
\|g_\psi(f)\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} |(\psi_{2^{-j}} * f)(x)|^2\right) dx = \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |(\psi_{2^{-j}} * f)(x)|^2 dx \\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} |\widehat{\psi}^2(2^{-j}\eta)\widehat{f}^2(\eta)| d\eta = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^{-j}\eta)|^2 |\widehat{f}|^2_{L^2(\mathbb{R}^n)} d\eta \\
&\leq C_\alpha^2 B^2 \|\widehat{f}\|^2_{L^2(\mathbb{R}^n)} = C_\alpha^2 B^2 \|f\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

□

## 4. The Boundedness of LPS Operator with an Improved Sufficient Condition

In this section, we will prove $L^p-$boundedness LPS operator with new condition

**Proposition 5.** *Let $1 < p < \infty$, If $\psi$ satisfies condition in Proposition 2 and (C3) condition then $g_\psi$ on $L^p(\mathbb{R}^n)$, that means there exists $C_{n,\alpha,p} > 0$ such that for $f \in L^p(\mathbb{R}^n)$ then*

$$\|g_\psi(f)\|_{L^p} \leq C_{n,\alpha,p}\|f\|_{L^p}$$

*Proof.* Using Proposition 3, 4 and by Marcinkiewicz interpolation Theorem, we obtain $g_\psi$ is bounded on $L^p(\mathbb{R}^n)$, where $1 < p \leq 2$. Let $g^*_\psi$ be the dual of $g_\psi$. It's assotiated with $\psi_{2^{-j}}(-x)$ by convolution relation, and implies (C1)-(C3) condition. By duality we obtain, for $2 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$\|g_\psi(f)\|_{L^p} = \sup_{\|h\|_{L^q}=1} \langle g_\psi(f), h\rangle = \sup_{\|h\|_{L^q}=1} \langle f, g^*_\psi(h)\rangle \leq \|f\|_{L^p} \sup_{\|h\|_{L^q}=1} \|g^*_\psi(h)\|_{L^q} \leq C_{n,\alpha,p}\|f\|_{L^p}.$$

□

We conclude the LPS operator is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

**Remark 6.** *For $\alpha \geq 1$ constant of boundedness is not depend with $\alpha$, but not for $0 < \alpha < 1$.*

## References


[1] Grafakos, L. (2014). *Clasical Fourier Analisis*, Springer Science+Busines Media, New York, 401–410 and 419–425.
[2] Hart, Jarot. (2013). *Bilinier Littlewood-Paley Square Function and Singular Integral*, University of Kansas
[3] Lerner dan Nazarov. (2015). *Intuitive dyadic calculus: the basic*, Cornell University Library, 7-10.
[4] Schlag, W. (2007). *A remark on Littlewood-Paley theory for the distorted Fourier transform*, Proceeding of the AMS.**135**(2), Washington, 437–451.
[5] Stein,E.M (1982). *The Development of Square Function in the Work of A.Zygmund*, Bulletin of the AMS. **7**(2), Washington, 359-376.
[6] Thoma, Eric. (2015). *Result on Fourier Multipliers*, University of Chicago, 1–16