Lebesgue-Hausdorff Line Integral of Hausdorff Measurable Multivariable Function over Simple Curve on [*a*,*b*]

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Abstract. Lebesgue Measure plays an important role in defining width of area under some graphs of real-valued function while the domain lies in real number system accurately. Yet such measure fails to approximate the area under the graph when we try to generalize the function with multiple variables. This is due to the Lebesgue measure has always zero value for any flat region lies in $\mathbb{R}^2$. In this paper we try to reconstruct more general line integral definition rather than usual Riemann line integral as well. The easiest way to do this is through the use of Hausdorff measure due to its dimension concept allows us to measure the length and area in $\mathbb{R}^2$ as well as it has been already done in $\mathbb{R}$. The result of this research is Lebesgue-Hausdorff line integral for Hausdorff measurable functions dimension 1 which lie under any simple curves.

Keyword: *Hausdorff measure, Lebesgue Hausdorff Line Integral, Riemann Line Integral.*

1. Backgrounds

As known previously, Hausdorff measure for some subset E of $\mathbb{R}^2$ is the supremum of total of some sets radius which cover E. This topic was mentioned by [1], i.e for any subset $E \subset \mathbb{R}^2$ and $\delta > 0$, δ −cover of E is some colletion of sets $\{I_k\}$ which cover E i.e $E \subset \bigcup_k I_k$ and diameter of each I_k is less than δ. For some $0 \le s \le 2$ Hausdorff content s dimension or shortly said is s-Hausdorff content with radius $\delta > 0$ is defined to be

$$H_\delta^s = \left\{ \sum_{k=1}^{\infty} |I_k|^s : \{I_k\} \ \delta - \text{cover } E \right\}$$

with $|I_k|$ denotes diameter I_k and s-Hausdorff measure is defined to be

$$H^s(E) = \lim_{\delta \to 0} H_\delta^s(E)$$

As smallest δ as possible will be given, it will give greater Hausdorff content. This fact implies following equivalent definition of Hausdorff measure as following

Theorem 1.1. $H^s(E) = \lim_{\delta\to 0} H^s_\delta(E) = \sup_{\delta>0} H^s_\delta(E)$.

Proof: If exist $\delta > 0$ such that $H^s_\delta(E) = \infty$ then it will be trivial. If for each $\delta > 0$, $H^s_\delta(E) < \infty$ then for each $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for every $0 < \delta < \delta_\varepsilon$ implies $\sup_{\delta>0} H^s_\delta(E) - \varepsilon < H^s_\delta(E)$. Therefore, $\sup_{\delta>0} H^s_\delta(E) - \varepsilon < H^s_\delta(E) \le \sup_{\delta>0} H^s_\delta(E) + \varepsilon$ for arbitrary $\varepsilon > 0$. In other words $\lim_{\delta\to 0} H^s_\delta(\mathrm{E}) = \sup_{\delta>0} H^s_\delta(E)$.

Furthermore, we have some useful following theorem

Theorem 1.2. Hausdorff measure satisfies following properties

- $\mathrm{H}^s(\phi) = 0$.
- *For each* $F \subset G \subset \mathbb{R}^2$ *and* $s > 0$ *satisfies* $H^s(F) \le H^s(G)$.
- *For each* $\{F_i\}$ is *a collection of subsets of* $\mathbb{R}^2$, *satisfies* $H^s\left(\bigcup_{i=1}^{\infty} F_i\right) \le \sum_{i=1}^{\infty} H^s(F_i)$.

Hausdorff measurability of set $E \subset \mathbb{R}^2$ is defined to be

$$H^S(A) = H^S(A \cap E) + H^S(A - E)$$

for every $A \subset \mathbb{R}^2$.

2. New Genelaizations

In the beginning of construction process, first we have to ensure that each curve with certain condition should be Hausdorff measurable. We mean such curve is a differentiable continuous function $\varphi: [a, b] \to \mathbb{R}^2$. Any curves defined on closed bounded interval subset of $\mathbb{R}$ always have finite length [2]. This will be explained as theorem below.

Theorem 2.1 *Each curve* $\varphi: [a, b] \to \mathbb{R}^2$ *has* $\int_a^b \|\varphi'(t)\| dt < \infty$.

Proof:

The concept of line integral is related strongly to the concept of length of the curve. Before we discuss about measurability we need the lemma below.

Lemma 2.2 *Let* $A \subset \mathbb{R}^2$ *be a set and mapping* $f: [a, b] \to A$. *If* f *is a continuous bijection and* A *is Hausdorff then* f is *homeomorphism*.

Proof: We just have to show that f^{-1} is continous. Let U be an arbitrary closed subset of $[a,b]$. We have the set U is compact. This implies $f(U)$ is compact in A. As A is a Hausdorff set and $f(U)$ is compact in $A \subset \mathbb{R}^2$ then $f(U)$ must be closed.

Lemma above gives direct consequence as corollary below.

Corollary 2.3. *Every image of injective curve* $\varphi: [a, b] \to \mathbb{R}^2$ *which I is bounded interval in* $\mathbb{R}$ *is homeomorphic with its domain*. Note that for each curve $\varphi: [a, b] \to \mathbb{R}^2$. Length of the curve is defined to be $L(\varphi) = \int_a^b \|\varphi'(t)\| dt$, with the norm is standard Euclidean norm on $\mathbb{R}^2$.

The following theorem explains a relation between injective curve and its length. The theorem plays an important role to decide an exact value of Haussdorff measure of curve image.

Theorem 2.4. *Every image of injective curve* $\varphi: [a,b] \to \mathbb{R}^2$ *is homeomorphic with interval* $[0, L(\varphi)]$.

Proof: We define map $f: [0. L(\varphi)] \to \varphi[a,b]$ i.e $f(u) = \varphi(x)$ if and only if $\int_a^x \|\varphi'(t)\| dt = u$ for some $u \in [0, L(\varphi)]$.

First, we prove that f is surjective. For each $\varphi(x) \in \varphi[a,b]$. If $x = a$ then it is obvious. For $x \neq a$. Since φ is rectifiable then $\int_a^x \|\varphi'(t)\| dt$ must be exist. Let us say $\int_a^x \|\varphi'(t)\| dt = u$ for some positive number u. Since we have $a < x \leq b$, hence

$$0 < u = \int_a^x \|\varphi'(t)\| dt \leq \int_a^b \|\varphi'(t)\| dt = L(\varphi).$$

Then $u \in [0, L(\varphi)]$ and $f(u) = \varphi(x)$.

We will show that f is injective. Let us say $\int_a^x \|\varphi'(t)\| dt = u$ and $\int_a^y \|\varphi'(t)\| dt = v$ so, $u \neq v$. The result is $x \neq y$. Since φ is injective, we have $\varphi(x) \neq \varphi(y)$. In other word, $f(u) \neq f(v)$.

Now we have to show that f is homeomorphism. For any $u \in [0, L(\varphi)]$ and any sequence $\{u_n\}$ in $[0, L(\varphi)]$ which is convergent to u. Let $\varepsilon > 0$ be any positive number, there is positive integer n_0 for every $n \geq n_0$ we have $|u_n - u| < \varepsilon$. This implies that each n, there will be x_n so, $\int_a^{x_n} \|\varphi'(t)\| dt = u_n$ and there is x so, $\int_a^x \|\varphi'(t)\| dt = u$.

Therefore

$$\begin{aligned}
\|f(u_n) - f(u)\| &= \|\varphi(x_n) - \varphi(x)\| \\
&\leq \left| \int_x^{x_n} |\varphi'(t)| dt \right| \\
&= \left| \int_x^a |\varphi'(t)| dt + \int_a^{x_n} |\varphi'(t)| dt \right| \\
&= \left| \int_a^{x_n} |\varphi'(t)| dt - \int_a^x |\varphi'(t)| dt \right| \\
&= |u_n - u| < \varepsilon .
\end{aligned}$$

In other words $\{f(u_n)\}$ converges to $f(u)$.

So, f is a continuous function. By lemma 1.2 we conclude that f is homeomorphism.

Next, we will find Hausdorff measure as an exact value for any curve defined on some interval $[a,b]$. First, we have to present following theorem as written in [1] below:

Theorem 2.5. *If* $A \subset \mathbb{R}^m$ *and* $f: \mathbb{R}^m \to \mathbb{R}^n$ *is Lipschitz then*

$$H^s(f(A)) \leq (Lip(f))^s H^s(A).$$

By $Lip(f)$ *is Lipschitz Constant of* f.

Proof: Let and fix some $\delta > 0$. Let $\{F_k\}$ be any δ-cover of A. This results $|f(F_k)| \leq Lip(f)|F_k|$ for each k. Therefore

$$H_\delta^s(f(A)) \leq \sum_{k=1}^{\infty} |f(F_k)|^s \leq Lip(f)^s \sum_{k=1}^{\infty} |F_k|^s .$$

Because of this condition works for any δ −cover of A, then we have

$$H_\delta^s(f(A)) \le Lip(f)H_\delta^s(A)\ .$$

Allowing δ tends to 0 gives

$$H^s(f(A)) \le Lip(f)H^s(A)\ .$$

Theorem above gives direct consequence as below.

Corollary 2.6. *If* $A \subset \mathbb{R}^m$ *and* $f:\mathbb{R}^m \to \mathbb{R}^n$ *is Lischitz then*

$$L^n(f(A)) \le (Lip(f))^s L^n(A).$$

Corollary above is easy to be proved by Isodiametric Inequality

[3] explained in his thesis the next theorem.

Theorem 2.7. *If* $\varphi: [a,b] \to \mathbb{R}^2$ *is injective curve then* $H^1(\varphi[a,b]) = L(\varphi)$.
By $L(\varphi) = \int_a^b \|\varphi'(t)\| dt$, *i.e arclength of curve* φ.

Proof: Let $\{[x_i, x_{i+1}]\}$ be any partition on $[a,b]$. We have

$$H^1(\varphi[x_i, x_{i+1}] = \lim_{\delta\to 0}\left(\inf \sum_{k=1}^{\infty} |F_k| : \{F_k\}\ \delta - \text{cover}\ \varphi[x_i, x_{i+1}] \right)$$
$$\ge \|\varphi(x_i) - \varphi(x_{i+1})\|$$

Since φ is injective then we have

$$H^1(\varphi[a,b]) \ge \sum_i \|\varphi(x_i) - \varphi(x_{i+1})\|$$

By taking supremum on the right side of equation results.

$$H^1(\varphi[a,b]) \ge L(\varphi)$$

Another side, let $f:[0, L(\varphi)] \to [a,b]$ by $f(u) = x$ if only if $\int_a^x \|\varphi'(t)\|\, dt = u$. In other words f is inverse function of primitive line integral φ along $[a, x]$. Therefore, $\|f(u) - f(v)\| \le |u - v|$. In other words f is Lipschitz. By theorem 2.5 we have

$$H^1(\varphi[a,b]) \le L(\varphi)$$

this ends our proof.

The last fact which is explained in theorem 2.4 and theorem 2.7 gives important result as follows

Corollary 2.8. *Every injective curve* $\varphi: [a,b] \to \mathbb{R}^2$ *is 1-Hausdorff measurable and*

$$H^1(\varphi[a,b]) = L(\varphi).$$

3. Hausdorff Measurable Functions

Note that a measure μ over $\mathbb{R}^2$ is said to be Borel measure if every open set in $\mathbb{R}^2$ is measurable [4]. Obviously Hausdorff measure is a Borel measure. This fact becomes a reason to define measurability of multivariable function as below.

Definition 3.1. *Let* $f: E \subset \mathbb{R}^2 \to \mathbb{R}$ *be a function. Function* f *is called s-Hausdorff measurable if for each open set* $O \subset R$ *implies* $f^{-1}(O)$ *s-Hausdorff measurable.*

The above definition is equivalent to the following theorem

Theorem 3.2. *Function $f: E \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is s-Hausdorff measurable iff for each $\alpha \in R$, the set $f^{-1}((\alpha,\infty))$ is s-Hausdorff measurable.*

Furthermore, the theorems implies that all of continuous function is s-Hausdorff measurable. Our next theorem is another consequence of the aforementioned definition

Theorem 3.3. *Let $f: E \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function. The following statements are equivalent:*

(i) *For each $\alpha \in R$ implies $f^{-1}((\alpha,\infty))$ s-Hausdorff measurable.*

(ii) *For each $\alpha \in R$ implies $f^{-1}([\alpha,\infty))$ s-Hausdorff measurable.*

(iii)*For each $\alpha \in R$ implies $f^{-1}((-\infty,\alpha))$ s-Hausdorff measurable.*

(iv)*For each $\alpha \in R$ implies $f^{-1}((-\infty,\alpha])$ s-Hausdorff measurable.*

Proof : Since $f^{-1}((\alpha,\infty))^c = f^{-1}((-\infty,\alpha])$ and $f^{-1}([\alpha,\infty))^c = f^{-1}((-\infty,\alpha))$ then (i) is equivalent to (iv) and statement (ii) is equivalent to (iii). We suffice to prove that (i) equivalent to (ii). Suppose that (ii) occurs. Therefore

$$
\begin{aligned}
f^{-1}((\alpha,\infty)) &= \{(x,y): f(x,y) > \alpha\} \\
&= \bigcup_{n=1}^{\infty}\left\{(x,y): f(x,y) \geq \alpha + \frac{1}{n}\right\} \\
&= \bigcup_{n=1}^{\infty} f^{-1}\left(\left[\alpha + \frac{1}{n}, \infty\right)\right)
\end{aligned}
$$

Since, $f^{-1}\left(\left[\alpha + \frac{1}{n}, \infty\right)\right)$ is s-Hausdorff measurable for each n then $f^{-1}([\alpha,\infty))$ s-Hausdorff measurable. Similarly, one can prove that $f^{-1}((-\infty,\alpha)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left(-\infty, \alpha - \frac{1}{n}\right]\right)$. Therefore (ii) implies (i). This completes the proof.

This is an important theorem for the construction of that integral

Theorem 3.4. *Let $\varphi: [a,b] \rightarrow \mathbb{R}^2$ be a curve and a function $f: E \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ so, the image φ lies in E. If f is 1-Hausdorff measurable then the restriction of f over image of curve φ is 1-Hausdorff measurable.*

Proof: Since f is 1-Hausdorff measurable, for each open set O we have $f^{-1}(O)$ is 1-Hausdorff measurable.

Note that the restriction f over image of curve φ, i.e $f|_{\varphi[a,b]}: \varphi[a,b] \rightarrow \mathbb{R}$ by $f|_{\varphi[a,b]}(x,y) = f(x,y)$ for each $(x,y) \in \varphi[a,b]$. Let O be an open set in $\mathbb{R}$. Because Hausdeorff measure is Borel measure (for arbitrary s), $f|_{\varphi[a,b]}^{-1}(O) = f^{-1}(O) \cap \varphi[a,b]$ should be 1-Hausdorff measurable. We conclude that φ is 1-Hausdorff measurable.

Here is the definition of characteristic function

Definition 3.4. *Let $E \subset \mathbb{R}^2$. Characteristic function of E is function such that*

$$\chi_E : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

by

$$\chi_E\,(x,y) = \begin{cases} 1, & (x,y) \in E \\ 0, & (x,y) \notin E \end{cases}$$

Thus,the definition is *s*-Hausdorff measurable if the domain is *s*-Hausdorff measurable. This fact is represented as following theorem

Theorem 3.5. *Let $E \subset \mathbb{R}^2$. The set E s-Hausdorff measurable if* f χ_E *s-Hausdorff measurable.*

Proof: Suppose $E \in \mathbb{R}$. Let $\alpha \geq 1$, then the set$\{ (x,y) \mid \chi_E\,(x,y) > \alpha\} = \emptyset$. If $0 < \alpha < 1$, then $\{ (x,y) \in \mathbb{R} \mid \chi_E\,(x,y) > \alpha\} = E$. If $\alpha < 0$, then $\{ (x,y) \in \mathbb{R} \mid \chi_E\,(x,y) > \alpha\} = \mathbb{R}^2$. Therefore, χ_E is *s*-Hausdorff measurable if f *E* *s*-Hausdorff measurable.

Note that the function $f, g: E \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be equal to almost everywhere if the set $\{(x,y): f(x,y) \neq g(x,y)\}$ has 0 Hausdorff measure. It is notated by $f = g$ a.e. The equality "almost everywhere" preserves measurability of Hausdorff.

4. Lebesgue-Hausdorff Line Integration

Before we ready to define the integral of multivariable function along with any simple curves, first we have to define the particular case of simple curve, i.e injective curve. We begin by definition of simple function as follow:

Definition 4.1 *Let E be an s-Hausdorff measurable set. An s-simple function is real valued function $\psi: E \subset \mathbb{R}^2 \to \mathbb{R}$ such that there is collection of s-Hausdorff measurable sets $\{E_1, E_2, \ldots, E_p\}$ such that $E = \bigcup_{k=1}^{p} E_k$ and collection of real numbers $a_1, a_2, \ldots, a_p$ with*

$$\psi(x) = \sum_{k=1}^{p} a_k \chi_{E_k}$$

Furthermore, the s-simple function is said to be simple function with canonical representation if for each index i, j such that $i \neq j$ implies $E_i \cap E_j = \emptyset$ and $a_i \neq a_j$.

Note that for special case if

As the image of a curve defined on compact interval is 1-Hausdorff measurable, we are able to define Lebesgue-Hausdorff integral of simple function along with injective curve as follows

Definition 4.2 *Let $\varphi: [a,b] \to \mathbb{R}^2$ be an injective curve and 1-simple function $\psi: \varphi[a,b] \to \mathbb{R}$ such that ψ has positive Hausdorff measure of its image on $\mathbb{R}^2$ and*

$$\psi = \sum_{k=1}^{p} a_k \chi_{E_k}$$

for some 1-Hausdorff measurable sets $\{E_1, E_2, \ldots, E_p\}$ by $\varphi[a,b] = \bigcup_{k=1}^{p} E_k$ and real numbers $a_1, a_2, \ldots, a_p$.

Lebesgue-Hausdorff line integral of ψ *is defined to be*

$$(LH)\int_{\varphi} \psi\, dH = \sum_{k=1}^{p} a_k H^1\,(E_k)$$

Therefore, integral must be exist due to theorem1.1 and every subset of a 1-Hausdorff measurable set which has the Hausdorff's dimension could not exceed beyond 1.
Now we are ready to define the integral of more general function as follows

Definition 4.3. (Lebesgue-hausdorff Line Integral) *Let* $\varphi: [a,b] \to \mathbb{R}^2$ *be an injective curve and function* $: \mathbb{R}^2 \to \mathbb{R}$.
Lebesgue-Hausdorff Lower Line Integral *is defined to be*:

$$(LH)\int_{\underline{\varphi}} f\, dH = sup\left\{(LH)\int_{\varphi} \zeta\, dH : \zeta \leq f, \zeta \text{ is } 1 - simplefunction\ on\ \varphi[a,b]\right\}$$

Lebesgue-Hausdorff Lower Line Integral is:

$$(LH)\int^{\overline{\varphi}} f\, dH = inf\left\{(LH)\int_{\varphi} \psi\, dH : f \leq \psi, \psi is\ 1 - simple\ function\ on\ \varphi[a,b]\right\}$$

Function f is called Lebesgue Hausdorff line integrable if

$$(LH)\int_{\underline{\varphi}} f\, dH = (LH)\int^{\overline{\varphi}} f\, dH < \infty$$

Furthermore the value of integral is defined to be

$$(LH)\int_{\varphi} f\, dH = (LH)\int_{\underline{\varphi}} f\, dH = (LH)\int^{\overline{\varphi}} f\, dH$$

Note that if $f: \mathbb{R}^2 \to \mathbb{R}$ is bounded and 1-Hausdorff measurable over image of φ, then the integral must be exist. For example of such functions are continuous functions and continous almost everywhere functions.
Now, we will extend the definition for more general type of curve. First we define another important type of curve. Such curve is called loop i.e the curve $\varphi: [a,b] \to \mathbb{R}^2$ which $\varphi(a) = \varphi(b)$ the restriction to (a,b) is an injection.

Note for any loop $\varphi: [a,b] \to \mathbb{R}^2$. It is easy to prove that $(a,b] = \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\left[a+\frac{m}{n}, b\right]$ with $\left[a+\frac{m}{n}, b\right] = \phi$ if $a+\frac{m}{n} > b$. Therefore $\varphi(a,b] = \varphi\left(\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\left[a+\frac{m}{n}, b\right]\right)$ with the restriction of φ over $\left[a+\frac{m}{n}, b\right]$ is injective if $\left[a+\frac{m}{n}, b\right] \neq \phi$. This implies $\varphi(a,b]$ is also 1-Hausdorff measurable. Since $\varphi[a,b] = \varphi(a,b] \cap \varphi(\{a\})^c$ and $H^1(\varphi(\{a\})) = 0$ hence $\varphi[a,b]$ is 1-Hausdorff measurable.

The fact above motivates to define Lebesgue-Hausdorff line integral for more general curve, i.e simple curve.

Definition 4.4. *A curve $\varphi: [a, b] \to \mathbb{R}^2$ is called simple if the restriction over (a, b) is an injection.*

From above definition is clear that injective curve and loop are examples of simple curves.

Definition 4.5. *Let $\varphi: [a, b] \to \mathbb{R}^2$ be a curve and 1-simple function $\psi: \varphi[a, b] \to \mathbb{R}$ such that ψ has positive Hausdorff measure of its image on $\mathbb{R}^2$ and*

$$\psi = \sum_{k=1}^{p} a_k \chi_{E_k}$$

for some 1-Hausdorff measurable collection of sets $\{E_1, E_2, \dots, E_p\}$ such that $\varphi[a,b] = \bigcup_{k=1}^{p} E_k$ and real numbers $a_1, a_2, \dots, a_p$.

Hausdorff-Lebesgue line integral ψ is defined to be

$$(LH)\int_{\varphi} \psi \, dH = \sum_{k=1}^{p} a_k H^1 (E_k)$$

Now we are ready to define Hausdorff-Lebesgue line integral for more general case as following

Definition 4.3.4. (Lebesgue-Hausdorff Line Integral) *Let $\varphi: [a, b] \to \mathbb{R}^2$ be a simple function and function $f: \mathbb{R}^2 \to \mathbb{R}$.*

Lebesgue-Hausdorff Lower Line Integral *is defined to be*:

$$(LH)\int_{\underline{\varphi}} f \, dH = sup\left\{(LH)\int_{\varphi} \zeta \, dH : \zeta \leq f, \zeta \text{ } is \text{ } 1 - simple \text{ } function \text{ } on \text{ } \varphi[a, b]\right\}$$

Lebesgue-Hausdorff Lower Line Integral is:

$$(LH)\int^{\overline{\varphi}} f \, dH = inf\left\{(LH)\int_{\varphi} \psi \, dH : f \leq \psi, \psi \text{ } is \text{ } 1 - simple \text{ } function \text{ } on \text{ } \varphi[a, b]\right\}$$

Function f is called Lebesgue-Hausdorff line integrable if

$$(LH)\int_{\underline{\varphi}} f \, dH = (LH)\int^{\overline{\varphi}} f \, dH < \infty$$

Furthermore the value of integral is defined to be

$$(LH)\int_{\varphi} f \, dH = (LH)\int_{\underline{\varphi}} f \, dH = (LH)\int^{\overline{\varphi}} f \, dH$$

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